Exact solution of the characteristic equation for transfer in the anisotropically scattering and absorbing medium

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A special form of anisotropic scattering phase function is shown to provide an exact solution of the characteristic equation for radiation transfer at depth within a scattering and absorbing medium. The solution is the Henyey-Greenstein function, the degree of extension of which depends on the albedo for single scattering and on the parameter of the phase function. Good applicability of the formulas obtained for a description of integral parameters of light fields in the seawater depth has been demonstrated.

I. Introduction

Exact solutions of the transfer equation play an important role in the study of the special features of light scattering in turbid media. On one hand, they can serve as test examples for developing programs for computational solution of this equation; on the other, they make it possible to obtain analytical connections between the apparent and inherent optical properties of the scattering medium.

There are few exact analytical solutions of the transfer equation which express the brightness distribution in the form of a compact formula. It is possible to mention, for example, the solution for the cases of isotropic and delta-function scattering as well as their combination-transport approximation. These solutions provide a good description of the limiting cases of isotropic and extremely anisotropic scattering, but they are hardly suitable for a precise description of phenomena actually existing in nature. This study' describes a new exact asymptotic solution of the transfer equation for an anisotropically scattering medium.

II. Derivation of Main Formulas

We proceed from the scaler equation of radiation transfer in a scattering and absorbing medium

$$\left(\mu d / dz + c\right) L(z,\mu,\phi) = \left(b / 4\pi\right) \int_{-1}^{1}\int_{0}^{2\pi} L(z,\mu',\phi') \times p(\cos\chi)\, d\mu' d\phi'. \tag{1}$$

Here $L(z,\mu,\phi)$ is the radiance at depth z, measured from the upper boundary of a semi-infinite layer; $\mu = \cos\theta$; θ is the zenith angle; $c = a + b$ is the attenuation coefficient; a and b are the absorption and scattering coefficients; $p(\cos\chi)$ is the scattering phase function calibrated according to condition

$$0.5\int_{0}^{\pi} p(\cos\chi)\sin\chi\, d\chi = 1 .$$

where $\cos\chi = \mu\mu' + \left[(1-\mu^2)(1-\mu'^2)\right]^{1/2}\cos(\phi-\phi')$, ϕ and ϕ' are the azimuthal angles, $\mu' = \cos\theta'$, χ is the scattering angle.

Taking into account the invariance under shift, we seek a solution of Eq. (1) at a depth within a scattering medium of the form

$$L(z,\mu,\phi) \equiv L(z,\mu) = L_0 \psi(\mu)\exp(-\gamma\tau), \tag{2}$$

where L_0 is determined from the boundary conditions, $\tau = c\,z$ is the optical depth, γ is the minimum eigenvalue[2] of Eq. (1) in terms of the absolute value ($c\gamma$ is the attenuation coefficient for totally diffuse light). Substituting Eq. (2) in Eq. (l),we obtain the characteristic equation of the transfer theory

$$\left(1-\gamma\mu\right)\psi(\mu) = \left(\omega_0 / 2\right)\int_{-1}^{1}\psi(\mu')\,\overline{p}(\mu,\mu')\,d\mu' \tag{3}$$

where $\omega_0 = b / c$ is the albedo for single scattering, $\overline{p}(\mu,\mu')$ is the aximuth-mean phase function,

$$\overline{p}(\mu,\mu') = (2\pi)^{-1}\int_{0}^{2\pi} p(\cos\chi)\,d\phi' .$$

Let us require the brightness distribution $\psi(\mu)$ to satisfy the standardization condition

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$$0.5\int_{-1}^{1}\psi(\mu)\,d\mu = 1\,.$$

Let us represent the phase function $p(\cos\chi)$, the azimuth mean phase function $\bar{p}(\mu,\mu')$, and the desired solution $\psi(\mu)$ in the form of Legendre polynomials series

$$p(\cos\chi) = \sum_{n=0}^{\infty} s_n\, P_n(\cos\chi)\,,\quad s_0 = 1\,, \tag{4}$$

$$\bar{p}(\mu,\mu') = \sum_{n=0}^{\infty} s_n\, P_n(\mu)\, P_n(\mu')\,, \tag{5}$$

$$\psi(\mu) = \sum_{n=0}^{\infty}(2n+1)\,\psi_n\, P_n(\mu)\,,\quad \psi_0 = 1\,. \tag{6}$$

Substituting Eqs. (5) and (6) in Eq. (3) and utilizing the recursion

$$(2n+1)\,\mu\, P_n(\mu) = n\, P_{n-1}(\mu) + (n+1)\, P_{n+1}(\mu)\,, \tag{7}$$

we can obtain recursion relations between coefficients ψ_n and s_n:

$$\psi_{n+1} = \left[(2n+1)\,\psi_n - \omega_0\, s_n\, \psi_n - \gamma\, n\,\psi_{n-1}\right]\big/\left[\gamma\,(n+1)\right], \tag{8}$$

$$\psi_0 - \gamma\,\psi_1 = \omega_0\,\psi_0\, s_0 \tag{9}$$

since $\psi_0 = s_0 = 1$, and

$$\psi_1 \equiv \eta = 0.5\int_{-1}^{1}\psi(\mu)\,\mu\, d\mu$$

is the mean cosine or asymmetry factor for the depth's radiance distribution $\psi(\mu)$, we get from Eq. (9) the familiar ratio

$$\gamma = (1-\omega_0)\,/\,\eta\,. \tag{10}$$

The objective of our study is to obtain any exact and compact analytical solution of Eq. (3) for the strongly anisotropic function $\bar{p}(\mu,\mu')$. Consequently, we should either cut off series (6) with respect to n or obtain a solution $\{\psi_n\}$, which is assembled into the analytic function. From Eq. (8) it is easy to see that the first procedure is impossible with any selection of s_n.

Let us examine a second approach. We shall attempt to proceed from the reverse. We shall find the function $\bar{p}(\mu,\mu')$ with which the solution of Eq. (3) will be a radiance distribution elongated into the depths of a scattering medium. It is convenient to select the Henyey-Greenstein function as such a distribution:

$$\begin{aligned}\psi(\mu) &= \left(1-\eta^2\right)\big/(1+\eta^2-2\,\eta\,\mu)^{3/2}\\ &= \sum_{n=0}^{\infty}(2n+1)\,\eta^n\, P_n(\mu)\,.\end{aligned} \tag{11}$$

Substituting in Eq. (8) $\psi_n = \eta^n$, we obtain

$$s_n = 2\,g\,n+1,\quad n>0\,, \tag{12}$$

where

$$g = (1+\omega_0)\,/\,(2\,\omega_0) - (1-\omega_0)\,/\,(2\,\omega_0\,\eta^2)\,. \tag{13}$$

By combining Eq. (12) and Eq. (5), we obtain the expansion

$$\begin{aligned}\bar{p}(\mu,\mu') &= g\sum_{n=0}^{\infty}(2n+1)\,P_n(\mu)\,P_n(\mu')\\ &\quad+(1-g)\sum_{n=0}^{\infty}P_n(\mu)\,P_n(\mu')\,.\end{aligned} \tag{14}$$

The first part on the right-hand side of Eq. (14) is proportional to Dirac's delta function, integrated in terms of the rule

$$\int_{-1}^{1}\delta(\mu-\mu')\,B(\mu')\,d\mu' = B(\mu)\,, \tag{15}$$

and having the following expansions in terms of Legendre's polynomials,[3]

$$\delta(\mu-\mu') = 0.5\sum_{n=0}^{\infty}(2n+1)\,P_n(\mu)\,P_n(\mu')\,, \tag{16}$$

$$\delta(1-\cos\chi) = 0.5\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\chi)\,. \tag{17}$$

It is easy to derive an expression for the second item in Eq. (14) by integrating the expansion of the generating function for Legendre polynomials

$$(1-2\,t\cos\chi+t^2)^{-1/2} = \sum_{n=0}^{\infty}t^n\, P_n(\cos\chi)\,, \tag{18}$$

over azimuthal angle ϕ,

$$\int_0^{2\pi}\left(1-2\,t\cos\chi+t^2\right)^{-1/2}d\phi = \sum_{n=0}^{\infty}t^n\, P_n(\mu)\,P_n(\mu')\,, \tag{19}$$

and taking the limit as $t\to 1$.

So we have

$$\begin{aligned}\bar{p}(\mu,\mu') &= 2g\,\delta(\mu-\mu') + \left[(1-g)\,/\,2\pi\right]\int_0^{2\pi}\left[2\,(1-\cos\chi)\right]^{-1/2}d\phi\\ &= (2\pi)^{-1}\int_0^{2\pi}\Big\{2\,g\,\delta(1-\cos\chi)\\ &\quad+(1-g)\left[2\,(1-\cos\chi)\right]^{-1/2}\Big\}\,d\phi.\end{aligned} \tag{20}$$

Consequently, the phase function $p(\cos\chi) = p_H(\cos\chi)$, where

$$p_H(\cos\chi) \equiv 2g\,\delta(1-\cos\chi) + (1-g)\left[2\,(1-\cos\chi)\right]^{-1/2}, \tag{21}$$

corresponds to the azimuth mean phase function (14).

Thus the radiance distribution (11) is the exact solution of Eq. (3) with phase function (21) which should be understood as a generalized function.[4] The parameter γ and the asymmetry factor η are expressed, as follows, through the inherent optical properties of the medium

$$\gamma = \left[(1-\omega_0)\,(1+\omega_0-2\,g\,\omega_0)\right]^{1/2}, \tag{22}$$

$$\begin{aligned}\eta &= \left[(1-\omega_0)/(1+\omega_0-2\,g\,\omega_0)\right]^{1/2} = \left[1+(4+2\sqrt{2}\,b_b)\big/a\right]^{-1/2}\\ &= \left\{(1-x)\,/\left[1+(3+2\sqrt{2})\,x\right]\right\}^{1/2},\end{aligned} \tag{23}$$

where $x = b_b\,/\,(a+b_b)$, $b_b = B\,b$ is the backward scattering coefficient;

$$B = 0.5\int_{-1}^{0}p_H(\mu)\,d\mu = (1-g)\,/\,(2+\sqrt{2}) \cong 0.2929\,(1-g)$$

is the backward scattering probability on phase function (21).

III. Integral Parameters of LightField

Let us calculate some integral parameters[5] of the light field in the depth of a scattering medium,

$$E_{0d} = 2\pi B_0 \int_0^1 \psi(\mu)\, d\mu\,, \quad E_{0u} = 2\pi B_0 \int_{-1}^0 \psi(\mu)\, d\mu\,, \qquad (24)$$

downward and upward scalar irradiances,

$$E_d = 2\pi B_0 \int_0^1 \psi(\mu)\, \mu\, d\mu\,, \quad E_u = -2\pi B_0 \int_{-1}^0 \psi(\mu)\, \mu\, d\mu\,, \qquad (25)$$

upward and downward irradiances caused by diffuse light, here $B_0 = L_0 \exp(-\gamma\, \tau)$,

$$\mu_d = E_d \,/\, E_{0d}\,, \quad \mu_u = E_u \,/\, E_{0u} \qquad (26)$$

average cosines of the depth's irradiance distribution for the lower and upper hemisphere,

$$R_0 = E_{0u} \,/\, E_{0d}\,, \quad R_H = E_u \,/\, E_d \qquad (27)$$

where R_H is the diffuse reflectance or irradiance ratio in the depth of the scattering layer.

Substituting Eq. (11) in Eqs. (24)-(27), we will get for μ_d, μ_u, scalar irradiance ratio R_0 and R_H,

$$\mu_d = (1+\eta^2)^{1/2} \left\{1 + \left[(1+\eta^2)^{1/2} - 1\right]\big/\eta\right\}\big/2, \qquad (28)$$

$$\mu_u = (1+\eta^2)^{1/2} \left\{1 - \left[(1+\eta^2)^{1/2} - 1\right]\big/\eta\right\}\big/2, \qquad (29)$$

$$R_0 = \left[(1-\eta)/(1-\eta)\right]\left[(1+\eta^2)^{1/2} - \eta\right], \qquad (30)$$

$$R_H = \left[(1-\eta)/(1-\eta)\right]\left[(1+\eta^2)^{1/2} - \eta\right]^2. \qquad (31)$$

For scattering media of the type of seawater, where x is small, we have

$$\eta = 1 - 3.414\, x\,, \quad x << 1\,, \qquad (32)$$

$$R_H = 0.293\, x\,, \quad x << 1. \qquad (33)$$

Figure 1 shows the dependences μ_d, μ_u, and R_H given by Eqs. (28), (29), and (31) as the functions of the asymmetry factor η. It is seen that the values obtained from Eqs. (28) and (31) provide a good description of the experimental data of Timofeyeva.[6]

IV. Comparison With Formulas of Other Studies

Let us represent Eq. (33) for the diffuse reflectances (DR) R_H as the function of the parameter $x = b_b \,/\, (a + b_b)$:

$$\begin{aligned} R_H = &\left[(1 + 5.8284\, x)^{1/2} - (1-x)^{1/2}\right] \\ &\times\left[(2 + 4.8284\, x)^{1/2} - (1-x)^{1/2}\right]\Big/ \\ &\times\left\{\left[(1 + 5.8284\, x)^{1/2} + (1-x)^{1/2}\right]\right. \\ &\left.\times\left[(2 + 4.8284\, x)^{1/2} - (1-x)^{1/2}\right]\right\}. \end{aligned} \qquad (34)$$

Although Eq. (34) is obtained from the exact solution of characteristic transfer Eq. (3) with phase function (21), its applicability for situations actualiy encountered in nature and especially for calculations of DR of sea depth requires verification. For this, let us compare the values of R_H produced by formula (34) with the values calculated by the Monte Carlo method in study.[7]

They approximate DR of the ocean by the expressions

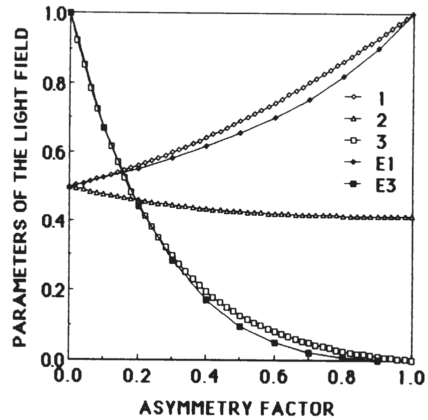

Fig. 1. Parameters of the light field in the depth of a scattering medium as a function of the asymmetry factor η: μ_d(l), μ_u(2), and R_H(3). The black marks are Timofeyeva's experimental data[6]: μ_d(E1) and R(E3)

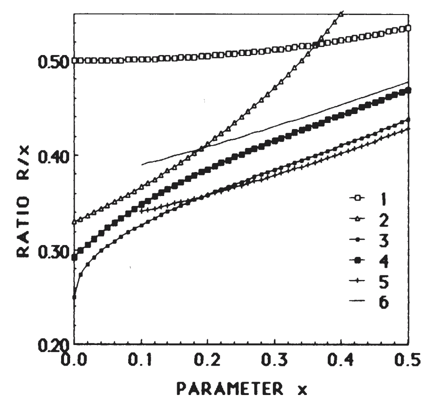

Fig. 2. Ratio $R \,/\, x$ $(R = R_i)$ as a function of the parameter x. The values R_i are calculated using the formulas 1–(37), 2–(38), 3–(39), 4–(34), 5–(35), 6–(36).

$$R_B = 0.0001 + 0.3244\, x + 0.1425\, x^2 + 0.1308\, x^3, \qquad (35)$$

$$R_G = 0.0003 + 0.3687\, x + 0.1802\, x^2 + 0.0740\, x^3. \qquad (36)$$

Equation (35) gives the values of the DR of the sea when its surface is illuminated by light directed to the nadir and Eq. (36), when its surface is illuminated by diffuse light. Figure 2 shows the values $R \,/\, x$ calculated with the use of Eqs. (35) and (36) and Eq. (34) (curves 5, 6, and 4). It is seen that throughout the range of validity of Eqs. (35) and (36), Eq. (34) gives a very good approximation and is usable for calculating the DR of the sea depth.

For comparison, the figure also shows the behavior of the values $R \,/\, x$ for the amounts of the DR $R = R_i$, calculated by the formulas of Gamburtsev,[8] Gurevich,[9] and Kubelka-Munk,[10]

$$R_K = \left\{a + b_b - \left[a\,(a + 2b_b)\right]^{1/2}\right\}\big/b_b \equiv \left[1 - \left(1 - x^2\right)^{1/2}\right]\big/x\,, \qquad (37)$$

Morel and Prieur,[11]

$$R_M = 0.33\, b_b \,/\, a \equiv 0.33\, x \,/\, (1-x)\,, \qquad (38)$$

and the author of this study,[12]

$$R_\infty = [(1-\bar{\mu})/(1+\bar{\mu})]^2,$$
$$\bar{\mu} = ((1-x)/\{1+2x+[x(4+5x)]^{1/2}\})^{1/2}. \qquad (39)$$

It is seen that Eqs. (34) and (39) provide a good description of the DR of the sea layer; Eq. (37) is rough, and Eq. (38) is applicable only when $x \le 0.2$.

V. Conclusions

(a) The equation for radiation transfer in the depth of a scattering medium (3) with an anisotropic phase function, given by Eq. (21), has an exact solution (11), which is valid throughout the whole range of variation in the inherent optical properties of the medium. (b) A brightness distribution (11) elongated into the depths of the medium makes it possible to obtain analytical expressions for the integral parameters of the light field in the depth of the scattering layer [Eqs. (28)—(31)]. (c) The formulas obtained in the study provide a good description of integral parameters of actual light fields in the seawater layer.

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