Self-consistent approach to the solution of the light transfer problem for irradiances in marine waters with arbitrary turbidity, depth, and surface illumination. I. Case of absorption and elastic scattering

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A self-consistent variant of the two-flow approximation that takes into account strong anisotropy of light scattering in seawater of finite depth and arbitrary turbidity is presented. To achieve an appropriate accuracy, this approach uses experimental dependencies between downward and total mean cosines. It calculates irradiances, diffuse attenuation coefficients, and diffuse reflectances in waters with arbitrary values of scattering, backscattering, and attenuation coefficients. It also takes into account arbitrary conditions of illumination and reflection from the bottom with the Lambertian albedo. This theory can be used for the calculation of apparent optical properties in both open and coastal oceanic waters, lakes, and rivers. It can also be applied to other types of absorbing and scattering medium such as paints, photographic emulsions, and biological tissues. © 1998 Optical Society of America

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1. Introduction

The majority of existing analytical methods for calculating light fields in scattering media is based on one or another variant of the two-flow approximation to the theory of radiation transfer.[1–12] The simplicity and convenience of the results that can be obtained with these approximations, and especially the possibility of one using them to solve inverse problems, favorably distinguish the two-flow theories from the potentially more accurate, but much less convenient, numerical methods for solving transport equations. This research is an extension of the earlier papers[13,14] by Haltrin and Kattawar. Two important enhancements are made here: the illumination of the sea surface is arbitrary (the light of the sky and the Sun) and the water layer has a finite bottom with the Lambertian albedo.

The major difference between this and previous two-flow approximations (see citations in Refs. 1–4) is threefold. First, it applies the two-flow equations only to the light scattered in all directions except forward; the unscattered and forward-scattered lights are regarded as light sources. Second, this approach uses parametric dependencies of the two-flow coefficients on inherent optical properties rather than making them equal to some constant. Third, this approach uses experimental data to establish relationships between two-flow coefficients and inherent optical properties. This theory results in a precise variant of the two-flow approximation that takes into account the strong anisotropy of the scattering and the asymmetry of the diffuse radiance field in seawater. The accuracy of the final equations in some cases approaches the precision of the numerical computations and matches the precision of in situ measurements.

There is a perception among some researchers that the two-flow approximation is imprecise and that its solutions are inadequate when compared with the similar values calculated from the solutions of the radiative-transfer equation. This is true only for poorly defined two-flow approximations. To show this, a system of two-flow equations for irradiances is derived below. This system is totally equivalent to the original radiative-transfer equation in the sense that both approaches give the same values of irradiances.

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Let us start from the one-dimensional radiative-transfer equation[15]:

$$\left(\mu \frac{d}{dz} + c\right)\tilde{L}(\varphi, \mu, z) = \frac{b}{4\pi}\int_{\Omega} p(\cos \gamma)\tilde{L}(\varphi', \mu', z)d\Omega'. \tag{1}$$

The system of coordinates here is chosen so that the x–y plane coincides with the outer boundary of the medium on which the radiation is incident (sea surface), whereas the 0–z axis is oriented into the medium (to the sea bottom). Here $\tilde{L}(z, \mu, \varphi)$ is the total radiance of the light; θ and φ are the zenith and azimuthal angles, respectively, that determine the direction of the light propagation, measured from the positive direction of the 0z axis; $c = a + b$ is the attenuation coefficient; a is the absorption coefficient; b is the scattering coefficient; $d\Omega' = \sin \theta' d\theta' d\varphi'$ is the element of the solid angle; $p(\gamma)$ is the light-scattering phase function; and γ is the light-scattering angle that is determined from the relation

$$\cos \gamma = \mu\mu' + \sqrt{1 - \mu^2}\sqrt{1 - \mu'^2}\cos(\varphi - \varphi'), \tag{2}$$

where $\mu = \cos \theta$, $\mu' = \cos \theta'$. The scattering phase function is normalized as follows:

$$\int_{4\pi} p(\cos \gamma)d\Omega' = 4\pi. \tag{3}$$

I now introduce the radiance and the scattering phase function averaged over the azimuthal angle φ:

$$\bar{L}(\mu, z) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{L}(\varphi, \mu, z)d\varphi, \tag{4}$$

$$\bar{p}(\mu, \mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\cos \gamma)d\varphi \equiv \bar{p}(\mu', \mu). \tag{5}$$

In terms of these values, the radiative transfer in Eq. (1), averaged over the azimuthal angle φ, is

$$\left(\mu \frac{d}{dz} + c\right)\bar{L}(\mu, z) = \frac{b}{2}\int_{-1}^{1} \bar{p}(\mu, \mu')\bar{L}(\mu', z)d\mu'. \tag{6}$$

I also introduce the following apparent optical properties of downward and upward irradiances:

$$\tilde{E}_d(z) = 2\pi \int_0^1 \bar{L}(\mu, z)\mu d\mu,$$
$$\tilde{E}_u(z) = -2\pi \int_{-1}^0 \bar{L}(\mu, z)\mu d\mu, \tag{7}$$

and of downward and upward spherical irradiances:

$$\tilde{E}_d^{\,0}(z) = 2\pi \int_0^1 \bar{L}(\mu, z)d\mu, \quad \tilde{E}_u^{\,0}(z) = 2\pi \int_{-1}^0 \bar{L}(\mu, z)d\mu. \tag{8}$$

The downward, upward, and total mean cosines are

$$\tilde{\mu}_d(z) = \frac{\tilde{E}_d(z)}{\tilde{E}_d^{\,0}(z)},$$
$$\tilde{\mu}_u(z) = \frac{\tilde{E}_u(z)}{\tilde{E}_u^{\,0}(z)},$$
$$\bar{\tilde{\mu}}(z) = \frac{\tilde{E}_d(z) - \tilde{E}_u(z)}{\tilde{E}_d^{\,0}(z) + \tilde{E}_u^{\,0}(z)}. \tag{9}$$

The diffuse reflection coefficient and the downward and upward diffuse attenuation coefficients at depth z are

$$R(z) = \frac{\tilde{E}_u(z)}{\tilde{E}_d(z)} \equiv \frac{\tilde{\mu}_u(z)[\tilde{\mu}_d(z) - \bar{\tilde{\mu}}(z)]}{\tilde{\mu}_d(z)[\tilde{\mu}_d(z) + \bar{\tilde{\mu}}(z)]},$$
$$k_d(z) = -\frac{1}{\tilde{E}_d(z)}\frac{d\tilde{E}_d(z)}{dz},$$
$$k_u(z) = -\frac{1}{\tilde{E}_u(z)}\frac{d\tilde{E}_u(z)}{dz}. \tag{10}$$

By applying operators $2\pi \int_0^1 d\mu \ldots$ and $2\pi \int_{-1}^0 d\mu \ldots$ to Eq. (6), we obtain the following exact system of two-flow equations with respect to the downward and upward irradiances:

$$\left[\frac{d}{dz} + \frac{c}{\tilde{\mu}_d(z)} - \frac{bg_d^{\,d}(z)}{2\tilde{\mu}_d(z)}\right]\tilde{E}_d(z) - \frac{bg_d^{\,u}(z)}{2\tilde{\mu}_u(z)}\tilde{E}_u(z) = 0,$$
$$-\frac{bg_u^{\,d}(z)}{2\tilde{\mu}_d(z)}\tilde{E}_d(z) + \left[-\frac{d}{dz} + \frac{c}{\tilde{\mu}_u(z)} - \frac{bg_u^{\,u}(z)}{2\tilde{\mu}_u(z)}\right]\tilde{E}_u(z) = 0. \tag{11}$$

The first careful derivations of the two-flow equations were performed by Zege[11] and Aas.[3] The coefficients $g_d^{\,d}$, $g_d^{\,u}$, $g_u^{\,d}$, and $g_u^{\,u}$ are given by the following equations and depend only on depth z:

$$g_d^{\,u}(z) = \frac{\int_0^1 d\mu \int_{-1}^0 d\mu' \bar{p}(\mu, \mu')\bar{L}(\mu', z)}{\int_{-1}^0 d\mu' \bar{L}(\mu', z)},$$

$$g_d^{\,d}(z) = \frac{\int_0^1 d\mu \int_0^1 d\mu' \bar{p}(\mu, \mu')\bar{L}(\mu', z)}{\int_0^1 d\mu' \bar{L}(\mu', z)},$$

$$g_u^{\,u}(z) = \frac{\int_{-1}^0 d\mu \int_{-1}^0 d\mu' \bar{p}(\mu, \mu')\bar{L}(\mu', z)}{\int_{-1}^0 d\mu' \bar{L}(\mu', z)},$$

$$g_u^{\,d}(z) = \frac{\displaystyle\int_{-1}^{0} \mathrm{d}\mu \int_{0}^{1} \mathrm{d}\mu' \bar{p}(\mu, \mu') \bar{L}(\mu', z)}{\displaystyle\int_{0}^{1} \mathrm{d}\mu' \bar{L}(\mu', z)}. \qquad (12)$$

For the case of isotropic scattering when $\bar{p}(\mu, \mu') = 1$, all four coefficients given by Eq. (12) are equal to 1.

If we know the depth behaviors of coefficients $g_d^{\,d}$, $g_d^{\,u}$, $g_u^{\,d}$, and $g_u^{\,u}$, mean cosines μ_d and μ_u, and inherent optical properties b and c, we can solve Eqs. (11) to find irradiances, and these solutions would be equivalent to the irradiances obtained by Eqs. (7) and (8) from the solutions of the radiative transfer in Eq. (1) or Eq. (6). However, in many cases we are not interested in taking this difficult way to find a solution and only want to solve a simplified version of Eqs. (11) by adopting some values for unknown functions $g_d^{\,d}$, $g_d^{\,u}$, $g_u^{\,d}$, and $g_u^{\,u}$ and μ_d and μ_u. In this case we arrived at a standard two-flow approach with all its shortcomings and degree of precision.

The two-flow approximation derived in this paper is constructed in such a way that it minimizes the losses in precision connected with the simplification of the precise system of Eqs. (11). The main objectives of the proposed theory are (1) to sustain applicability of all derived equations for any possible combination of inherent optical properties b and c, i.e., to make it applicable for all types of water, including very clean oceanic and extremely turbid coastal waters; and (2) to keep the precision of the derived equations in the same range as those obtained by the contemporary in situ optical probes, i.e., in the range of 10–15%.

To achieve this the underwater light was divided into unscattered and scattered components. The light scattered in the narrow range of the forward direction merged with the unscattered light. The propagation of the unscattered and forward-scattered component was treated according to the Bouguer law. The two-flow equations were derived for the rest of the scattered light (without the forward-scattered component that merged with the unscattered component).

2. Approach

We start from the exact Eqs. (11) for the irradiances that are derived from the scalar equation for transfer (1). To make Eqs. (11) solvable it is necessary, however, to approximate the resulting coefficients of the system of the two-flow equations. We use two main steps to reduce the exact, but analytically unresolvable, system of Eqs. (11) to an approximate system that can be resolved easily. The first step consists of replacing the initial arbitrary phase function with the transport phase function. This greatly simplifies equations, but introduces an excessive error. We reclaim the lost accuracy, in the next step, by introducing empirical relationships between the downward cosine μ_d and the total mean cosine $\bar{\mu}$ derived from laboratory and in situ data.[16–18] This relationship shows that with the change of the Gordon *et al.* parameter[19] $g = B\omega_0/(1 - \omega_0 + B\omega_0)$ between 0 and 1 (here $\omega_0 = b/c$ is the single-scattering albedo and $B = 0.5 \int_{\pi/2}^{\pi} p(\cos\gamma)\sin\gamma\, d\gamma$ is the backscattering probability), the total mean cosine $\bar{\mu}$ also varies between 0 and 1 and the downward mean cosine $\bar{\mu}_d$ decreases from 1 to 0.5.

The main purpose of this research is to obtain equations that relate inherent optical to apparent optical properties for any input radiance distribution. These equations are derived to be valid in the complete range of variability of optical properties of natural water.

In the transfer theory, requirements of both simplicity and precision are mutually exclusive. For a successful resolution of the problem, therefore, we accepted a compromise by determining the degree of simplicity and precision.

A. Background

We refer to our method as the self-consistent method. For a better understanding of the idea of the method, we quote an example from classical mechanics,[20] from which it was adopted. Suppose we have to obtain the equation of motion of a material body around some center of attraction. The law of attraction is unknown, or it is known only partially. In addition, we have some information on the shape of trajectories in the form of dependencies between the integral parameters of these trajectories. This problem can be solved if we use the available information to constrain the acceptable solutions. In this example knowledge of additional information on consequences (trajectory parameters) makes it possible to compensate for the lack of information on causes (attraction forces).

In the theory of radiative transfer the main causes are the inherent optical properties such as the scattering law characteristics (volume-scattering function and single-scattering albedo), and the main consequences are the apparent optical properties, such as the angular distribution of radiance, as functions of depth. As a rule, the volume-scattering function is only approximately known, with unknown precision.

Thus, in our attempts to solve the problem of light-field calculation in a scattering and absorbing medium, we restrict ourselves to the simplest transport approximation of the volume-scattering function. The information that is lost through this simplification can be restored by accepting functional dependencies between the integral parameters of the radiance angular distribution, which are derived from an approximation of the experimental data.

B. Formulation of the Approximate Method

We now start from the scalar Eq. (1) and describe the transport of optical radiation in a sea with depth z_B. In anisotropically light-scattering media such as seawater, the scattering phase function $p(\cos\gamma)$ has a distinct diffraction peak[21] near $\gamma = 0$. The light rays that are scattered in a small solid angle near the forward direction ($\gamma \approx 0$) form the halo part of the scattered light.[22] They are, for all practical purposes, indistinguishable from the unscattered rays. This suggests that the halo part of the rays should not be considered as scattered light. It means that the forward diffraction peak should be eliminated from the scattering phase function.

We separate the main part of the halo rays by representing the scattering phase function as a sum of isotropic and anisotropic parts:

$$p(\cos\gamma) = 2B + (1 - 2B)p_\delta(\cos\gamma), \tag{13}$$

$$p_\delta(\cos\gamma) = \frac{p(\cos\gamma) - 2B}{1 - 2B}, \quad \int_{4\pi} p_\delta(\cos\gamma)\mathrm{d}\Omega' = 4\pi, \tag{14}$$

where $\delta(x)$ is the Dirac delta function[23] and B is the probability of backscattering. With the elongation of the scattering phase function, the following relation is valid[24]:

$$\lim_{B\to 0} p_\delta(\cos\gamma) = 2\delta(1 - \cos\gamma) \equiv 4\pi\delta(\varphi - \varphi')\delta(\mu - \mu'). \tag{15}$$

Next, we introduce the following auxiliary scattering phase function:

$$\tilde{p}(\cos\gamma) = 2B + 2(1 - 2B)\delta(1 - \cos\gamma), \quad \int_{4\pi} \tilde{p}(\cos\gamma)\mathrm{d}\Omega' = 4\pi. \tag{16}$$

By substituting $p(\cos\gamma)$ with its equivalent $\tilde{p}(\cos\gamma) + [p(\cos\gamma) - \tilde{p}(\cos\gamma)]$, we can rewrite the radiative-transfer equation in Eq. (1) in the following equivalent form:

$$\left(\mu \frac{\partial}{\partial z} + \alpha\right)\tilde{L}(z, \mu, \varphi) = \frac{b_B}{2\pi}\int \tilde{L}(z, \mu', \varphi')\mathrm{d}\Omega' + \frac{b}{4\pi} \times \int [p(\cos\gamma) - \tilde{p}(\cos\gamma)] \times \tilde{L}(z, \mu', \varphi')\mathrm{d}\Omega', \tag{17}$$

where $b_B = bB$ is the backscattering coefficient and $\alpha = a + 2b_B$ is the renormalized attenuation coefficient.

Let $L_q(\mu, \varphi)$ be the radiance of light just below the sea surface, i.e., at $z = +0$. Next, let $L(z, \mu, \varphi)$ be the radiance of the scattered component minus forward-scattered rays. We represent $\tilde{L}(z, \mu, \varphi)$, the total radiance at depth z, as a sum of scattered light in all directions except forward, $L(z, \mu, \varphi)$, and a component $L_q(\mu, \varphi)\exp(\alpha z/\mu)$, which is a sum of direct and forward-scattered light:

$$\tilde{L}(z, \mu, \varphi) = L(z, \mu, \varphi) + \begin{cases} L_q(\mu, \varphi)\exp(-\alpha z/\mu), & 0 < \mu \le 1 \\ 0, & -1 \le \mu \le 0 \end{cases}. \tag{18}$$

The coefficient $\alpha = a + 2b_B$ introduced here can be regarded as the beam attenuation coefficient for the sum of the unscattered and forward-scattered light.

On the right-hand side of Eq. (18) we did not include the light reflected from the bottom. By doing this, we assume that either the sea layer is optically thick ($\alpha z_B \gg 1$) or its lower boundary reflects light according to Lambert's law.

After substituting Eq. (18) into the exact Eq. (17), we obtain the following equation for the radiance of scattered light:

$$\left(\mu \frac{\partial}{\partial z} + \alpha\right)L(z, \mu, \varphi) = \frac{b_B E^0(z) + q(z, \mu, \varphi) + \Delta(z, \mu, \varphi)}{2\pi}, \tag{19}$$

where $E^0(z)$ is the scalar irradiance by diffuse light:

$$E^0(z) = \int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} L(z, \mu', \varphi')\mathrm{d}\mu', \tag{20}$$

and $q(z, \mu, \varphi)$ is the source function:

$$q(z, \mu, \varphi) = \frac{b}{2}\int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} p(\cos\gamma)L_q(\mu', \varphi') \times \exp(-\alpha z/\mu')\mathrm{d}\mu' - 2\pi b(1 - 2B) \times L_q(\mu, \varphi)\exp(-\alpha z/\mu), \tag{21}$$

$$\Delta(z, \mu, \varphi) = \frac{b}{2}\int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} [p(\cos\gamma) - \tilde{p}(\cos\gamma)]L(z, \mu', \varphi')\mathrm{d}\mu'. \tag{22}$$

The function Δ, given by Eq. (22), describes all the differences in scattering between real seawater with phase function $p(\cos\gamma)$ and the simple analytically resolvable model with transport scattering function $\tilde{p}(\cos\gamma)$. Equation (19) is completely equivalent to Eq. (1). The approximation that uses the function $\Delta(z, \mu, \varphi)$ in Eq. (19) corresponds to inclusion halo rays in the nonscattered light radiance.

Note that the value of Δ, given by Eq. (22), vanishes in the following two limiting cases: (a) for the isotropic scattering when $p(\cos\gamma) = 1$ and (b) for the extremely anisotropic scattering with the delta-shaped phase function $p(\cos\gamma) = 2\delta(1 - \cos\gamma)$.

3. Equations for Irradiances

Because Eq. (19) includes an arbitrary phase function $p(\cos\gamma)$, it cannot be solved analytically. By dropping the difference between the real and the transport phase function (Δ) we can reduce our problem to the case of exactly solvable isotropic scattering. However, an ellipsoidally shaped radiance distribution, obtained in this case in the depth of the scattering medium, poorly describes the experimental and in situ results of Refs. 16–18. In the next step, by dropping the difference Δ between the real and the transport phase functions, we can take into account its effect in an indirect manner.

We seek a solution of the problem given by Eq. (19) in the two-flow approximation by formally setting $\Delta = 0$ and assuming that the radiance distribution of scattered light within the ocean is described by the following asymptotic formula:

$$L(z, \mu) \propto L^{\infty}(\mu)\exp(-\kappa c z),$$

$$L^{\infty}(\mu) = \frac{(1-\bar{\mu}^2)^2}{(1-\bar{\mu}\mu)^3}, \quad \frac{1}{2}\int_{-1}^{1} L^{\infty}(\mu)\mathrm{d}\mu = 1. \quad (23)$$

We derived this angular radiance distribution $L^{\infty}(\mu)$ from the experimental results of Refs. 16–18. In relation (23) the value κ is the parameter of the deep regime, and $k_a = \kappa c$ is the diffuse attenuation coefficient in the depth when the asymptotic light regime is established.[25–27] The value $\bar{\mu}$ is the mean cosine over the normalized radiance distribution $L^{\infty}(\mu)$ given by relation (23), i.e.,

$$\bar{\mu} = \frac{1}{2}\int_{-1}^{1} L^{\infty}(\mu)\mu\mathrm{d}\mu. \quad (24)$$

Relation (23), if we apply it to the total radiance, is valid only in the asymptotic depth regime, which means that it is not applicable in the areas near the surface and bottom where the asymptotic regime is not established. These intermediate layers have vertical dimensions of the order of the optical depth or c^{-1}. When relation (23) is applied to the scattered portion of light, its range of validity expands and the size of the intermediate layers decreases. Diffuse illumination of the surface and Lambertian-type reflection of the bottom also improve the applicability of relation (23).

The approach used here has an analogy in classical mechanics.[20] It corresponds to replacing unknown forces by known constraints (the method of the Lagrange multipliers). The force corresponds to Δ and the constraints correspond to the adoption of the radiance distribution given by relation (23).

To derive two-flow equations we introduce irradiances E_i and scalar irradiances E_i^0 for the diffuse light from above ($i = 1$) and from below ($i = 2$) by the following equations:

$$E_1(z) = \int_0^{2\pi}\mathrm{d}\varphi\int_0^1 L(z, \mu, \varphi)\mu\mathrm{d}\mu,$$

$$E_2(z) = -\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^0 L(z, \mu, \varphi)\mu\mathrm{d}\mu, \quad (25)$$

$$E_1^0(z) = \int_0^{2\pi}\mathrm{d}\varphi\int_0^1 L(z, \mu, \varphi)\mathrm{d}\mu,$$

$$E_2^0(z) = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^0 L(z, \mu, \varphi)\mathrm{d}\mu. \quad (26)$$

Here index $i = 1$ corresponds to the downwelling irradiances and index $i = 2$ corresponds to the upwelling irradiances, i.e., $E_1 \equiv E_d$, $E_2 \equiv E_u$, $E_1^0 \equiv E_d^0$, and $E_2^0 \equiv E_u^0$. The average downward $\mu_1(z)$ and upward $\mu_2(z)$ cosines are defined according to the formulas

$$\mu_1(z) \equiv \mu_d(z) = \frac{E_1(z)}{E_1^0(z)}, \quad \mu_2(z) \equiv \mu_u(z) = \frac{E_2(z)}{E_2^0(z)}. \quad (27)$$

We apply the radiance distribution given by relation (23) to Eqs. (27). After simple integration, the following formulas for downward and upward mean cosines are obtained:

$$\mu_1(z) \to \mu_1 \equiv \mu_d = \frac{1}{2-\bar{\mu}},$$

$$\mu_2(z) \to \mu_2 \equiv \mu_u = \frac{1}{2+\bar{\mu}}. \quad (28)$$

Note that, according to the experiments,[16–18] the relation $\mu_d = 1/(2-\bar{\mu})$ is satisfied with high accuracy for the asymptotic radiance distributions inside modeled scattering and absorbing media and in the sea. It is valid for arbitrary values of a, b, and b_B if the asymptotic or depth regime is established.

We use Eq. (19) with $\Delta = 0$ and integral operators

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 \mu\mathrm{d}\mu \ldots, \quad \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^0 \mu\mathrm{d}\mu \ldots, \quad (29)$$

then apply Eqs. (25)–(27) and replace average cosines $\mu_i(z)$ by the values given by Eqs. (28). As a result, the following matrix equation for the downward and upward irradiances E_1 and E_2 is obtained:

$$\hat{L}_{ik}E_k(z) = f_i(z). \quad (30)$$

The differential matrix operator $\hat{L}_{ik}$ in Eq. (30) has the following form:

$$\hat{L}_{ik} = \begin{Vmatrix} \frac{\mathrm{d}}{\mathrm{d}z} + (2-\bar{\mu})(a+b_B) & -(2+\bar{\mu})b_B \\ -(2-\bar{\mu})b_B & -\frac{\mathrm{d}}{\mathrm{d}z} + (2+\bar{\mu})(a+b_B) \end{Vmatrix}. \quad (31)$$

The source functions f_1 and f_2 on the right-hand side of Eq. (30) can be expressed by the scattering phase function $p(\cos\gamma)$ and the radiance distribution just below the sea surface:

$$f_1(z) = b\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 \mathrm{d}\mu[2B - \psi(\mu)] \times L_q(\mu, \varphi)\exp(-\alpha z/\mu), \quad (32)$$

$$f_2(z) = b\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 \mathrm{d}\mu\psi(\mu) \times L_q(\mu, \varphi)\exp(-\alpha z/\mu), \quad (33)$$

where

$$\psi(\mu) = \frac{1}{2}\int_0^1 \bar{p}(-\mu', \mu)\mathrm{d}\mu',$$

$$\bar{p}(\mu, \mu') \equiv \frac{1}{2\pi}\int_0^{2\pi} p(\cos\gamma)\mathrm{d}\varphi. \tag{34}$$

In Eq. (30) and elsewhere in this paper the repeated indices imply summation: $u_i v_i \equiv \sum_i u_i v_i$.

The negative eigenvalue of the system of the two-flow equation in Eq. (30) is given by the formula

$$-\alpha_\infty = \bar{\mu}(a + b_B) - [4a(a + 2b_B) + \bar{\mu}^2 {b_B}^2]^{1/2}. \tag{35}$$

On the other hand, the exact eigenvalue for both Eqs. (1) and (19) is determined by Gershun's formula[28]:

$$\alpha_\infty = \kappa c \equiv \frac{a}{\bar{\mu}}. \tag{36}$$

Now let us accept the fact that the diffuse attenuation coefficients given by Eqs. (35)–(36) match, i.e., we force the radiance distribution for scattered light to match the experimental distribution given by Eqs. (23). In this case it is possible to find α_∞ and $\bar{\mu}$ as functions of the inherent optical parameters of the medium a and b_B. We can now solve Eqs. (35) and (36) relative to $\bar{\mu}$. The result is the following formula that connects the mean cosine with the absorption and backscattering coefficients a and b_B [or Gordon's parameter $g = b_B/(a + b_B)$]:

$$\begin{aligned}\bar{\mu} &= \left\{\frac{a}{a + 3b_B + [b_B(4a + 9b_B)]^{1/2}}\right\}^{1/2} \\ &= \left\{\frac{1-g}{1 + 2g + [g(4 + 5g)]^{1/2}}\right\}^{1/2} \\ &\equiv \left\{\frac{1 + 2g - [g(4 + 5g)]^{1/2}}{1+g}\right\}^{1/2}.\end{aligned} \tag{37}$$

From Eqs. (36) and (37) it is easy to obtain the equation for the deep regime parameter $\kappa = \alpha_\infty/c$ (i.e., the depth diffuse attenuation coefficient in the units of the beam attenuation coefficient c):

$$\begin{aligned}\kappa &= (1 - \omega_0) \\ &\quad\times\left(1 + \frac{3B\omega_0 + \{B\omega_0[4(1 - \omega_0) + 9B\omega_0]\}^{1/2}}{1 - \omega_0}\right)^{1/2} \\ &\equiv (1 - \omega_0)\left\{\frac{1+g}{1 + 2g - [g(4 + 5g)]^{1/2}}\right\}^{1/2},\end{aligned} \tag{38}$$

where $\omega_0 = b/c \equiv b/(a + b)$ is the single-scattering albedo.

4. Solutions for Irradiances

We now look for a solution to Eq. (30) as a sum of the general (the first two terms) and partial (the last term) solutions[23]:

$$\begin{aligned}E_1(z) &= Aa_1 \exp(-\alpha_\infty z) + Ee_1 \exp(\alpha_0 z) \\ &\quad + \int_0^{z_B} G_{1k}(z - z')f_k(z')\mathrm{d}z', \\ E_2(z) &= Aa_2 \exp(-\alpha_\infty z) + Ee_2 \exp(\alpha_0 z) \\ &\quad + \int_0^{z_B} G_{2k}(z - z')f_k(z')\mathrm{d}z'.\end{aligned} \tag{39}$$

where

$$\alpha_0 = \bar{\mu}(a + b_B) + [4a(a + 2b_B) + \bar{\mu}^2 {b_B}^2]^{1/2} \tag{40}$$

is the second eigenvalue of Eq. (30), and $a_1 = e_2 = 1$, $a_2 = R_\infty$, and $e_1 = R_0$, where R_∞ and R_0 are given by the following equations:

$$\begin{aligned}R_\infty &= \frac{(2 - \bar{\mu})b_B}{(2 + \bar{\mu})(a + b_B) + \alpha_\infty} \\ &= \frac{(2 - \bar{\mu})(a + b_B) - \alpha_\infty}{(2 - \bar{\mu})b_B} \equiv \left(\frac{1 - \bar{\mu}}{1 + \bar{\mu}}\right)^2,\end{aligned} \tag{41}$$

$$\begin{aligned}R_0 &= \frac{(2 + \bar{\mu})b_B}{(2 - \bar{\mu})(a + b_B) + \alpha_0} \\ &= \frac{(2 + \bar{\mu})(a + b_B) - \alpha_0}{(2 - \bar{\mu})b_B} = \left(\frac{2 + \bar{\mu}}{2 - \bar{\mu}}\right)R_\infty.\end{aligned} \tag{42}$$

Below we show that R_∞ is the diffuse reflectance coefficient of an optically infinitely deep sea measured far below its surface or illuminated by diffuse light.

The constants A and E in the solutions given by Eqs. (39) are determined by the boundary conditions. Green's matrix $G_{ik}(z)$ in Eqs. (39) satisfies the following matrix differential equation:

$$\hat{L}_{il}G_{lk}(z) = \delta_{ik}\delta(z), \tag{43}$$

where δ_{ik} is the Kronecker symbol (or the unity matrix). After some algebra it is possible to show that Green's matrix has the following form:

$$\begin{aligned}G_{ik}(z) &= \begin{Vmatrix}1 & R_0 \\ R_\infty & R_0R_\infty\end{Vmatrix}\frac{H(z)\exp(-\alpha_\infty z)}{1 - R_0R_\infty} \\ &\quad + \begin{Vmatrix}R_0R_\infty & R_0 \\ R_\infty & 1\end{Vmatrix}\frac{H(-z)\exp(\alpha_0 z)}{1 - R_0R_\infty},\end{aligned} \tag{44}$$

where $H(z)$ is the Heaviside (or step) function:

$$H(z) = \begin{cases}1, & z > 0, \\ 0, & z \le 0.\end{cases} \tag{45}$$

First we substitute Eq. (44) into Eqs. (39) and then impose the following two-boundary conditions at the levels of the sea surface ($z = 0$) and the sea bottom ($z = z_B$):

$$E_1(0) = E_0, \quad E_2(z_B) = A_B[E_1(z_B) + E_1^{\,f}(z_B)], \quad (46)$$

where A_B is the Lambertian albedo of the bottom and $E_1^{\,f}$ is the total downward irradiance produced by the direct and forward-scattered light:

$$E_1^{\,f}(z) = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 L_q(\mu, \varphi)\exp(-\alpha z/\mu)\mu \mathrm{d}\mu,$$
$$E_2^{\,f}(z) = 0. \quad (47)$$

As a result, the following equations for descending and ascending irradiances of diffuse light were obtained:

$$E_1(z) = [E_0 + M(z)]\exp(-\alpha_\infty z) + R_0 N(z)[\exp(\alpha_0 z) - \exp(-\alpha_\infty z)], \quad (48)$$

$$E_2(z) = R_\infty[E_0 + M(z)]\exp(-\alpha_\infty z) + N(z)[\exp(\alpha_0 z) - R_0 R_\infty \exp(-\alpha_\infty z)], \quad (49)$$

where

$$M(z) = \frac{1}{1 - R_0 R_\infty} \int_0^z \mathrm{d}z'\{[f_1(z') + R_0 f_2(z')]\exp(\alpha_\infty z') - R_0[R_\infty f_1(z') + f_2(z')]\exp(-\alpha_0 z')\}, \quad (50)$$

$$N(z) = \frac{A_B - R_\infty}{R_0 \Delta_B}[E_0 + M(z_B)]\exp(-\nu z_B) + \frac{1}{1 - R_0 R_\infty} \times \int_0^{z_B} \mathrm{d}z'[R_\infty f_1(z') + f_2(z')]\exp(-\alpha_0 z') + \frac{A_B}{R_0 \Delta_B} \times \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 B_q(\mu, \varphi)\exp[-(\alpha_0 + \alpha/\mu) z_B]\mu \mathrm{d}\mu, \quad (51)$$

$$\Delta_B = \frac{(1 - R_0 A_B)}{R_0}[1 + R_0 \xi_B \exp(-\nu z_B)],$$
$$\xi_B = \frac{A_B - R_\infty}{1 - R_0 A_B}, \quad (52)$$

$$\nu = \alpha_0 + \alpha_\infty = \frac{2a}{\bar{\mu}}\left(1 - \frac{\bar{\mu}^2}{1 - g}\right) = \alpha_\infty \frac{7 + 2\bar{\mu}^2 - \bar{\mu}^4}{3 - \bar{\mu}^2}. \quad (53)$$

For the totally diffuse illumination of the sea surface we choose the simplest way to solve the problem. That is, assume that the irradiance of light from external sources passing through the upper boundary,

$$E_q^{\,e} = E_1^{\,f}(0) = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 L_q(\mu, \varphi)\mu \mathrm{d}\mu, \quad (54)$$

is completely diffuse. Then we take it into account simply with the help of the boundary condition $E_1(0) = E_q^{\,0}$ and by simultaneously setting external sources $f_i(z) = 0$. On this basis we make the following substitutions:

$$E_0 = E_q^{\,0}, \quad M(z) = 0, \quad N(z) = E_q^{\,0} \frac{A_B - R_\infty}{R_0 \Delta_B} \exp(-\nu z_B). \quad (55)$$

In this particular case, Eqs. (48) and (49) were transformed into the following solutions for the downward and the upward diffuse irradiances:

$$E_d(z) \equiv E_1(z) = E_q^{\,0} \frac{(1 - R_0 A_B)}{\Delta_B R_0}\{1 + R_0 \xi_B \times \exp[-\nu(z_B - z)]\}\exp(-\alpha_\infty z), \quad (56)$$

$$E_u(z) \equiv E_2(z) = E_q^{\,0} \frac{(1 - R_0 A_B)}{\Delta_B R_0}\{R_\infty + \xi_B \times \exp[-\nu(z_B - z)]\}\exp(-\alpha_\infty z). \quad (57)$$

5. Diffuse Attenuation Coefficients for Diffuse Illumination

We now calculate downward [$k_d \equiv k_1 = -\mathrm{d} \ln E_1(z)/\mathrm{d}z$] and upward [$k_u \equiv k_2 = -\mathrm{d} \ln E_2(z)/\mathrm{d}z$] diffuse attenuation coefficients for irradiances. Substitution of Eqs. (56) and (57) in these definitions yields the following results:

$$k_d(z) \equiv k_1(z) = \frac{a}{\bar{\mu}} \frac{1 - \eta R_0 \xi(z)}{1 - R_0 \xi(z)},$$
$$k_u(z) \equiv k_2(z) = \frac{a}{\bar{\mu}} \frac{R_\infty - \eta \xi(z)}{R_\infty - \xi(z)}, \quad (58)$$

where

$$\xi(z) = \xi_B \exp[-\nu(z_B - z)], \quad -1 \le \xi(z) \le 1, \quad (59)$$

$$\eta = \frac{g - R_\infty}{g - R_0}, \quad 4/3 \le \eta \le 3. \quad (60)$$

The formulas for R_∞, R_0, ν, and $\bar{\mu}$ are given by Eqs. (41), (42), (53), and (37). The values for k_1 and k_2 computed by use of Petzold phase functions[29] can be obtained from published computations for μ_u and μ_d in Table 1 of Ref. 12.

Figure 1 illustrates the behavior of the parameters $\bar{\mu}$, $\mu_d \equiv \mu_1$, R_∞, and R_0 as functions of Gordon's parameter $g = B\omega_0/(1 - \omega_0 + B\omega_0)$.

6. Transmission and Diffuse Reflection Coefficients

We now calculate the transmission coefficient of the layer (0–z) for diffuse light $T(z) = E_d(z)/E_d(0)$ and the diffuse reflectance coefficient $R(z) = E_u(z)/E_d(z)$ of the shallow sea. By use of Eqs. (56) and (57) it was easy to obtain the following equations:

$$T(z) = \frac{1 + R_0 \xi(z)}{1 + R_0 \xi(z)\exp(-\nu z)} \exp(-\alpha_\infty z), \quad (61)$$

$$R(z) = \frac{R_\infty + \xi(z)}{1 + R_0 \xi(z)}, \quad \xi(z) = \frac{A_B - R_\infty}{1 - R_0 A_B} T_\nu,$$
$$T_\nu = \exp[-\nu(z_B - z)], \quad (62)$$

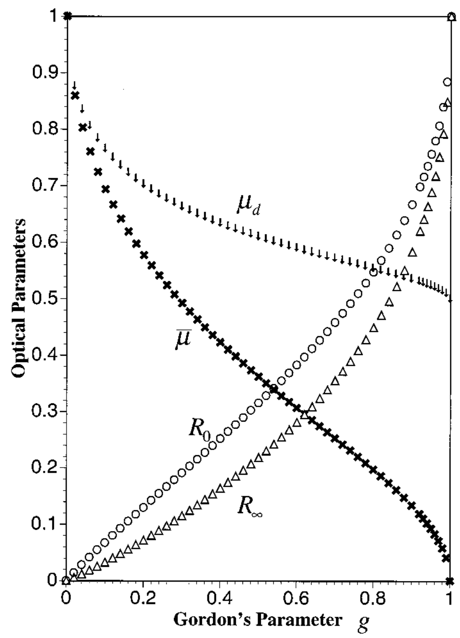

Fig. 1. Behavior of the optical parameters $\bar{\mu}$, $\mu_d \equiv \mu_1$, R_∞, and R_0 as functions of Gordon's parameter $g = B\omega_0/(1 - \omega_0 + B\omega_0)$.

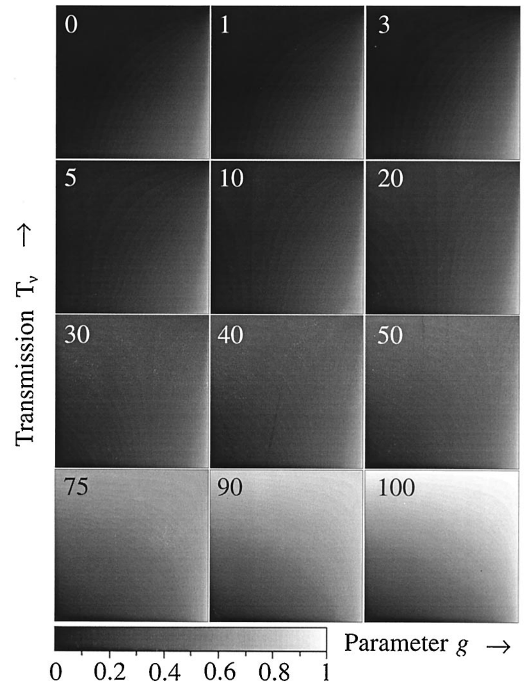

Fig. 2. Diffuse reflectance of the sea with the Lambertian albedo plotted as a function of Gordon's parameter g and the transmission $T_v = \exp[-\nu(z_B - z)]$ for different bottom albedos. The A_B numbers are given in percent in the upper left corners of the density plots.

where ν is given by Eq. (53). Figure 2 displays 12 density plots of the diffuse reflection coefficients $R(z)$ as a function of Gordon's parameter g and the transmission T_ν for a set of bottom albedos A_B.

By setting $z = 0$ in Eqs. (62), we can obtain the diffuse reflection coefficient of an ocean with depth z_B and the Lambertian bottom albedo A_B. The result is

$$R = \frac{R_\infty + \xi_B \exp(-\nu z_B)}{1 + R_0 \xi_B \exp(-\nu z_B)}, \quad \xi_B = \frac{A_B - R_\infty}{1 - R_0 A_B}. \quad (63)$$

Equations (63) generalize the formula for the diffuse reflection coefficient known as Kubelka–Munk[5,30] for ocean-type absorption and anisotropically scattering media with a reflecting bottom.

The limit at $\nu z_B \to \infty$ in Eqs. (63) gives us $R \to R_\infty$. Consequently, the value R_∞ corresponds to the diffuse reflectance of the optically very deep sea illuminated with diffuse light. The formula for R_∞ as a function of inherent optical properties a and b_B is

$$R_\infty = \left(\frac{1 - \bar{\mu}}{1 + \bar{\mu}}\right)^2, \ \bar{\mu} = \left\{\frac{a}{a + 3b_B + [b_B(4a + 9b_B)]^{1/2}}\right\}^{1/2}. \quad (64)$$

Equations (64) can be used for computation of the diffuse reflectances of deep seas with arbitrary turbidity without any restrictions on the values of b_B and a. The values of the diffuse reflectance coefficient computed with Eqs. (64) are close to the exact values of the diffuse reflectance coefficient calculated for the delta-hyperbolic scattering phase function.[31]

7. Asymptotics

We now investigate the asymptotics of Eqs. (64) for small and large ratios of the backscattering coefficient to the absorption coefficient b_B/a.

In the limit when $b_B/a \to 0$, i.e., when absorption predominates, for the average cosine we have

$$\bar{\mu} = 1 - \left(\frac{b_B}{a}\right)^{1/2} + \frac{3}{8}\left(\frac{b_B}{a}\right)^{3/2} - \frac{1}{8}\left(\frac{b_B}{a}\right)^2 - \frac{39}{128}\left(\frac{b_B}{a}\right)^{5/2}, \quad \frac{b_B}{a} \ll 1, \quad (65)$$

and for the diffuse reflection coefficient,

$$R_\infty = \frac{1}{4}\frac{b_B}{a} + \frac{1}{4}\left(\frac{b_B}{a}\right)^{3/2} - \frac{3}{32}\left(\frac{b_B}{a}\right)^{5/2} + \frac{5}{64}\left(\frac{b_B}{a}\right)^3, \quad \frac{b_B}{a} \ll 1. \quad (66)$$

The numerical analysis shows that the first term in Eq. (66) varies from Eq. (35) in Ref. 3 by Aas, calculated with the Petzold phase functions,[21,29] in the range of 15%.

For the media with low absorption we have another asymptotic formula:

$$R_\infty = \left(1 - 2\sqrt{\frac{a}{6b_B}}\right) \Big/ \left(1 + 2\sqrt{\frac{a}{6b_B}}\right), \quad \frac{b_B}{a} \gg 1. \quad (67)$$

In the case of isotropic scattering, Eq. (67) coincides with the asymptotically correct formula derived by Gate.[32]

8. Arbitrary Illumination

After examining the conditions for reflection and transmission of light by the upper boundary,[33] we determined the angular distribution of the brightness created by external sources. We found the source functions f_i from Eq. (32). At the same time, Eqs. (48) and (49), in which $E_0 = 0$ [because of the boundary condition $E_1(0) = 0$], give the solution to the problem. This approach is also suitable for the case of combined illumination of the surface by direct and diffuse light.

For the case of arbitrary illumination of the surface of an optically infinitely thick ocean, the diffuse reflection coefficient is derived from Eqs. (48) and (49) at $z = 0$:

$$R = (1 - \bar{\mu})^2 \int_0^{2\pi} d\varphi \int_0^1 \times \left[1 + \frac{2\bar{\mu}}{1 + \bar{\mu}^2} \frac{\psi(\mu) - B}{B}\right] \frac{J(\mu, \varphi)\mu d\mu}{1 + \mu\bar{\mu}(4 - \bar{\mu}^2)}, \quad (68)$$

where $J(\mu, \varphi) \equiv E_q(\mu, \varphi)/E_q^{\,0}$ is the normalized distribution of the light radiance transmitted through the sea surface and the function ψ is given by Eqs. (34).

In the presence of direct Sun illumination in the directions (μ_s, φ_s), the normalized irradiance is given by the formula

$$J_s(\mu, \varphi) \equiv \frac{1}{\mu_s} \delta(\varphi - \varphi_s)\delta(\mu - \mu_s). \quad (69)$$

Substituting Eq. (69) for J_s into Eq. (68), for the diffuse reflection of the sea illuminated by the direct sunlight we obtain

$$R_s = \frac{(1 - \bar{\mu})^2}{1 + \mu_s\bar{\mu}(4 - \bar{\mu}^2)} \left[1 + \frac{2\bar{\mu}}{1 + \bar{\mu}^2} \frac{\psi(\mu_s) - B}{B}\right]. \quad (70)$$

When the phase function is isotropic in the backward hemisphere, the function ψ is equal to the probability of backscattering, i.e., $\psi(\mu_s) = B$. In this case we obtained the following simplified variant of Eq. (70):

$$R_s = \frac{(1 - \bar{\mu})^2}{1 + \mu_s\bar{\mu}(4 - \bar{\mu}^2)}, \quad (71)$$

where

$$\mu_s = [1 - (\sin Z_S/n_w)^2]^{1/2} \equiv [1 - (\cos h_s/n_w)^2]^{1/2} \quad (72)$$

is the cosine of the angle at which the direct sunlight enters the sea, Z_S is the Sun zenith angle, $h_s = 90° - Z_S$ is the Sun elevation angle, and $n_w \approx 1.34$ is the refractive index of seawater.

We now consider the case of combined illumination of the sea surface by the direct light of the Sun and the diffuse skylight. By use of Eq. (68) it is easy to obtain the equation for the diffuse reflection coefficient of the optically deep sea:

$$R_c = \frac{R_\infty + sR_s}{1 + s}, \quad s = \frac{E_s}{E_0}. \quad (73)$$

Here R_∞ and R_s are given by Eqs. (64) and (71); E_s and E_0 are, respectively, the downward irradiances from direct sunlight and from diffuse light in the sky. Both values are measured just below the sea surface.

The quantity s depends on the transmission by the sea surface[34] and the optical parameters of the atmosphere.[35]

9. Inverse Problem

The inverse problem according to Tao *et al.*[12] consists of calculating the absorption and backscattering coefficients of seawater from the remotely measured reflectance of the ocean. The remote reflectance of the ocean that is due to the averaging over large areas of the sea with different surface wave patterns is proportional to the diffuse reflectance of the ocean. The diffuse reflection of the ocean that is measured near the surface is given by Eq. (70). In the majority of important remote-sensing cases, the difference between the surface diffuse reflection coefficient of the sea and the diffuse reflection coefficient of the same sea measured in its depth are insignificant, i.e. this difference lies in the range of the precision of the measurements.[36–38] In this case the inverse problems of calculating inherent optical properties a and b_B of the sea are reduced to the reverse of Eqs. (64). By resolving this equation, we can obtain the following formula for the absorption coefficient:

$$a = b_B\Phi_H(R_\infty), \quad (74)$$

$$\Phi_H(R_\infty) = \frac{(1 - \sqrt{R_\infty})^2(1 + 4\sqrt{R_\infty} + R_\infty)}{4R_\infty}. \quad (75)$$

Comparison with the numerical computations by the Monte Carlo method shows that the function $\Phi_H(R_\infty)$ is more precise[4] than with the Kubelka–Munk function[5]:

$$\Phi_K = (1 - R_\infty)^2/(2R_\infty). \quad (76)$$

It is especially true for smaller diffuse reflectances $R_\infty \leq 0.5$, typical to open and clean coastal ocean waters.

10. Validation

To validate the presented theory, an extensive set of Monte Carlo calculations[39] of the apparent optical properties in the broad range of inherent optical properties a, b, and $p(\cos\gamma)$ was conducted.[37] As input parameters, all 15 Petzold phase functions[21,29] with the associated absorption and scattering coefficients were used. Each run consisted of ten million histories. The Sun elevation angle h_s varied from 0° to 90° in 2.5° steps. The total number of runs was 555, and each run produced outputs for 401 depth intervals. Because the self-consistent approach can be used to calculate direct light precisely and scattered light approximately

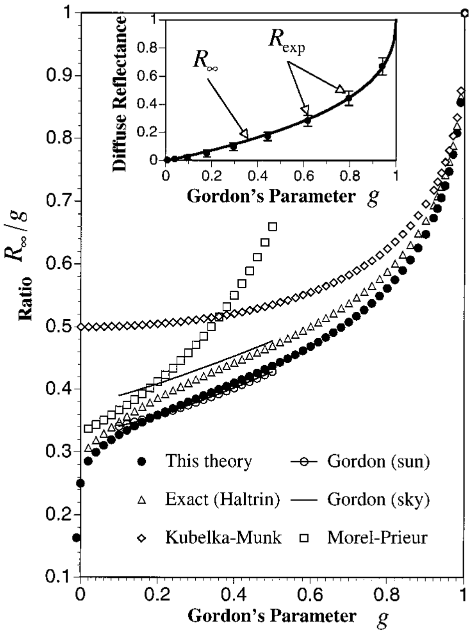

Fig. 3. Ratio R/g of the diffuse reflectance coefficients to Gordon's parameter g plotted as a function of g. The values of R/g computed with Eqs. (64) are compared with the similar values computed by other approaches (see the formulas in Appendix A). The inset shows the diffuse reflectance coefficient of the optically deep sea as a function of parameter g. The curve was plotted by use of Eqs. (64). The filled circles represent the experimental values of Timofeyeva.[40]

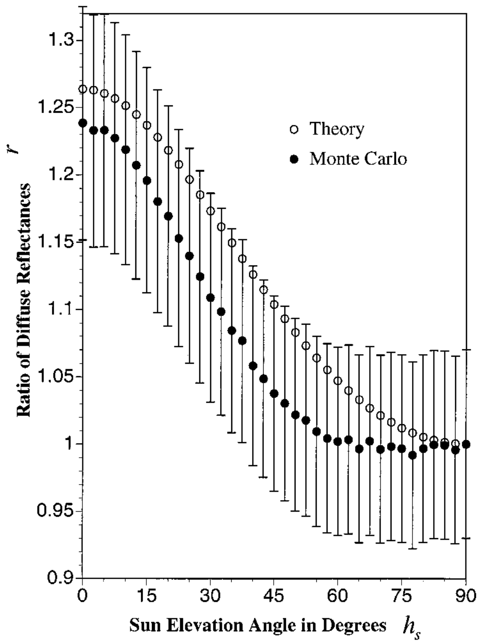

Fig. 4. Ratio of the diffuse reflectances $r = R_s(h_s)/R_s(90°)$ plotted as a function of the Sun elevation angle h_s. The open circles show the values calculated by use of Eq. (71). The filled circles represent the similar values computed with the Monte Carlo procedure for the Petzold phase function number 14 (see Refs. 37 and 39).

with Eqs. (23), the largest errors for irradiances are located inside the two layers near the sea surface and the sea bottom. The overall differences between the numerical values and the values computed by the presented equations were always less than 14.7%.

One of the most important parameters calculated here is the diffuse reflection coefficient of an optically infinitely thick ocean. This parameter is given by Eqs. (64). The values of the diffuse reflectance coefficient computed with the theory and the experimental values measured by Timofeyeva[40] match with the precision of 10% for the entire range of inherent optical properties (see the inset in Fig. 3). Figure 3 displays the values of $R_\infty(g)/g$ as a function of Gordon's parameter g. The values of $R_\infty(g)/g$ in Fig. 3 were computed according to Eqs. (64), formulas proposed by other investigators (see Appendix A), and values computed with the Monte Carlo simulation. Figure 3 shows the advantage of Eqs. (64) in comparison with the Kubelka–Munk and Morel–Prieur[41] formulas. The differences between theoretical and simulated values are less than 7% for the presented case.

Comparison of the calculated values with this theory and computed with the Monte Carlo simulation diffuse reflectance coefficients of the sea illuminated by the direct light of the Sun is shown in Fig. 4. The errors between values calculated with this approach and simulated values are in the range of 12% for all computed cases.

One of the most practically important results for optical oceanography is shown in Eqs. (64), which connect the diffuse reflection coefficient R_∞ with the backscattering and absorption coefficients b_B and a. Equations (64) are more complex than the widely overused equations $R_\infty = k_0 b_B/a$ or $R_\infty = k_0 b_B/(a + b_B)$, $0.2 \le k_0 \le 0.35$, but they are valid in the entire range of variability of optical properties b_B and a. Figure 5 shows the results of the sensitivity analysis of this equation. The shaded area displays 10,000 values of R_∞ computed with Eqs. (64) for randomly generated $\bar{\mu}$ with the simulated error in the range of $\pm 15\%$. The filled boxes in Fig. 5 show the experimental values of R_∞ taken from Ref. 42. This figure shows an almost perfect fit between the theoretical formula [Eqs. (64)] and the experimental data. The relative error of computing R_∞ with Eqs. (64) depends on the relative errors of b_B and a through the following equation:

$$\frac{\Delta R_\infty}{R_\infty} = k_g\left(\frac{\Delta b_B}{b_B} - \frac{\Delta a}{a}\right), \quad k_g(g) = -\frac{4g(1-g)}{1-\bar{\mu}^2}\frac{\partial \bar{\mu}}{\partial g}. \quad (77)$$

The functional dependence of the coefficient k_g is shown in the inset of Fig. 5. The coefficient k_g is less than 1.0 in almost all cases except a very small region near $g = 0$ where it reaches maximum $k_g \approx 1.03$. This means that, in the worst case, when Δb_B and Δa have different signs, we should measure b_B and a with at least 7.5% error to achieve 15% precision for R_∞.

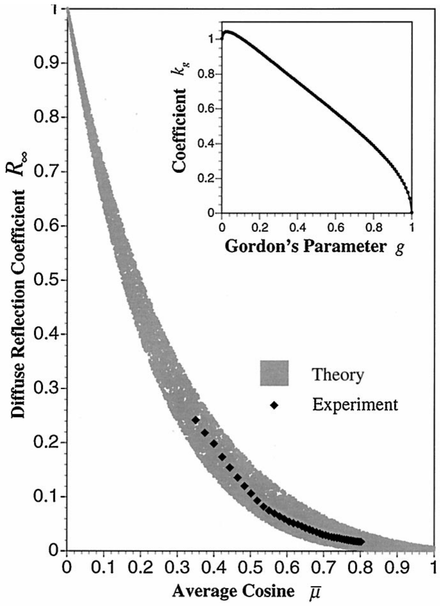

Fig. 5. Comparison of theoretical values of the diffuse reflection coefficient given by Eqs. (64) with the experimental values published in Ref. 42. The inset shows the dependence of coefficient k_g in Eqs. (77) on parameter g.

11. Conclusions

The self-consistent variant of the two-flow approximation presented here is good for any values of the inherent optical parameters of seawater. The equations obtained here give more accurate values for the apparent optical properties than the Kubelka–Munk and other well-known theories. This approach calculates irradiances, diffuse attenuation coefficients, and diffuse reflectance coefficients in waters with arbitrary scattering and absorption coefficients, arbitrary conditions of illumination, and the sea bottom with the Lambertian albedo. This theory can be used successfully for computations of apparent optical properties in open and coastal oceanic waters, lakes, and rivers.

Appendix A: Formulas Used in Fig. 3

The following formulas proposed in different publications were used to display ratios of R/g in Fig. 3.

The diffuse reflection coefficient for the direct Sun illumination at nadir was computed with the Monte Carlo simulation by Gordon *et al.*[19]:

$$R \equiv R_{Gs} = 0.0001 + 0.3244g + 0.1425g^2 + 0.1308g^3, \quad (0.1 \le g \le 0.5). \tag{A1}$$

The diffuse reflection coefficient for the diffuse illumination was computed with the Monte Carlo simulation by Gordon *et al.*[19]:

$$R \equiv R_{Gd} = 0.0003 + 0.3687g + 0.1802g^2 + 0.0740g^3, \quad (0.1 \le g \le 0.5). \tag{A2}$$

The diffuse reflection coefficient was calculated according to the theories of Gamburtsev,[6] Kubelka and Munk,[5] Sagan and Pollack,[7] and Coackley and Chylek[8] [Eq. (13) of Ref. 8 with $\tau_1 \to \infty$]:

$$R \equiv R_{\mathrm{KM}} = \frac{a + b_B - \sqrt{a(a + 2b_B)}}{b_B} = \frac{1 - \sqrt{1 - g^2}}{g} = \frac{\sqrt{1+g} - \sqrt{1-g}}{\sqrt{1+g} + \sqrt{1-g}}. \tag{A3}$$

The empirical diffuse reflection coefficient was calculated according to Morel and Prieur[41]:

$$R \equiv R_{\mathrm{MP}} = 0.33\,\frac{b_B}{a} = \frac{0.33g}{1-g}, \quad (0 \le g \le 0.2). \tag{A4}$$

The exact diffuse reflection coefficient was calculated for the delta-hyperbolic phase function of scattering according to Haltrin[31]:

$$R \equiv R_H = \left(\frac{1-h}{1+h}\right)(h - \sqrt{1+h^2})^2,$$

$$h = \left[\frac{a}{a + (4 + 2\sqrt{2})b_B}\right]^{1/2} \equiv \left[\frac{1-g}{1 + (3 + 2\sqrt{2})g}\right]^{1/2}. \tag{A5}$$

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