Diffuse reflection coefficient of a stratified sea

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A differential equation of a Riccati type for the diffuse reflection coefficient of a stratified sea is proposed. For a homogeneous sea with arbitrary inherent optical properties this equation is solved analytically. For an inhomogeneous sea it is solved approximately for any arbitrary stratification. The resulting equation expresses the diffuse reflection coefficient of the sea through vertical profiles of absorption and backscattering coefficients, bottom albedo, and sea depth. The results of calculations with this equation are compared with Monte Carlo computations. It was found that the precision of this approach is in the range of 15%.

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1. Introduction

The ocean optics community now makes extensive use of the following relationships between the diffuse reflection coefficient of a homogeneous optically infinite deep sea and the inherent optical properties of seawater:

$$R = k\,\frac{b_B}{a}, \quad \text{or} \quad R = k\,\frac{b_B}{a + b_B}, \quad \frac{b_B}{a} \le 0.3, \qquad (1)$$

where a is the absorption coefficient, b_B is the backscattering coefficient, and k is a numerical coefficient that varies from 0.2 to 0.5. Several versions of Eqs. (1) were derived with the two-flow approximation of the radiation transfer theory.[1–3] The applicability of the radiative transfer theory to ocean optics problems has been verified by numerous authors in ground truth measurements and in model experiments.[2,3] Therefore it is possible to accept that the radiative transfer theory can be applied successfully in calculations of oceanic light fields.

However, real marine waters are inhomogeneous. Typically, the optical properties of natural waters are vertically stratified. This optical stratification varies depending on area, time of year, and hydrologic conditions. Consequently, it is important to derive an expression for the diffuse reflection coefficient of the sea suitable for practical use with arbitrary vertical profiles of the optical characteristics. Such an expression is proposed here with emphasis placed on a general solution for stratified water.

It is well known that modern numerical methods[3,4] allow us to calculate the apparent optical properties of stratified sea with any given precision. At the same time we should not underestimate the value of simple analytical solutions on the basis that they are not precise enough. There are several reasons to use them: (a) Calculations with analytical equations are ~100,000 times faster than exact numerical calculations tuned to the same precision. (b) It is possible to derive custom equations for any specific stratification model and to play with these equations with simple programmable calculators. (c) The physics of phenomenon is more easily understood with analytic equations than with results from numerical analysis.

2. Basic Equations

We start with the system of the two-flow equations for renormalized light irradiances[5] obtained in Refs. 6 and 7:

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + (2 - \bar{\mu})(a + b_B)\right] E_d(z) - (2 + \bar{\mu}) b_B E_u(z) = 0,$$

$$-(2 - \bar{\mu}) b_B E_d(z) + \left[-\frac{\mathrm{d}}{\mathrm{d}z} + (2 + \bar{\mu})(a + b_B)\right] \times E_u(z) = 0, \qquad (2)$$

where $E_d(z)$ and $E_u(z)$ are the downward and upward irradiances, respectively; z is the depth coordinate (the Cartesian axis $0z$ is normal to the sea surface and directed to the bottom); $\bar{\mu}$ is the average cosine over the deep water angular distribution of radiance[7];

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$b_B = bB$ is the backscattering coefficient; b is the scattering coefficient;

$$B = 0.5 \int_{\pi/2}^{\pi} p(\cos\theta)\sin\theta d\theta \quad (3)$$

is the backscattering probability; and $p(\cos\theta)$ is the phase function of scattering[8] to the angle θ.

The system of equations (2) was derived from the radiative transfer equation under the following assumptions:

(1) The scattering phase function was represented in the transport approximation as the sum of isotropic and δ-shaped phase functions: $p(\cos\gamma) = 2B + 2(1 - B)\delta(1 - \cos\gamma)$, where $\delta(1 - \cos\gamma)$ is the Dirac delta function.

(2) The deep water angular radiance distribution, used in calculations of the downward and upward average cosines μ_d and μ_u, was represented by a relation that closely approximates the results observed by Timofeyeva[9,10]:

$$L_\infty(\mu) = \frac{(1 - \bar{\mu}^2)^2}{(1 - \bar{\mu}\mu)^3}, \quad \bar{\mu} = \frac{\int_{-1}^{1} L_\infty(\mu)\mu d\mu}{\int_{-1}^{1} L_\infty(\mu) d\mu}, \quad \mu = \cos\theta. \quad (4)$$

The upward and downward mean cosines over radiance distribution (5) are equal to

$$\bar{\mu}_d = \int_0^1 L_\infty(\mu)\mu d\mu \bigg/ \int_0^1 L_\infty(\mu) d\mu = \frac{1}{2 - \bar{\mu}}, \quad (5)$$

$$\bar{\mu}_u = -\int_{-1}^0 L_\infty(\mu)\mu d\mu \bigg/ \int_{-1}^0 L_\infty(\mu) d\mu = \frac{1}{2 + \bar{\mu}}. \quad (6)$$

(3) The average cosine $\bar{\mu}$ was expressed through the inherent optical properties of media a and b_B as follows: (a) By first solving Eqs. (2) in the lower half-space, we obtained a relation for the diffuse attenuation coefficient in the asymptotic regime,[1,6,7]

$$\alpha_\infty = [4a(a + 2b_B) + \bar{\mu}^2 b_B^{\,2}]^{1/2} - \bar{\mu}(a + b_B). \quad (7)$$

(b) Then by jointly solving Eq. (7) with the exact Gershun relation, $\alpha_\infty = a/\bar{\mu}$, we found the average cosine $\bar{\mu}$, which is expressed as a function of the absorption and backscattering coefficients, a and b_B, or of Gordon's parameter g ($0 \le g \le 1$):

$$\bar{\mu} = \left\{\frac{a}{a + 3b_B + [b_B(4a + 9b_B)]^{1/2}}\right\}^{1/2} \equiv \left[\frac{1 - g}{1 + 2g + [g(4 + 5g)]^{1/2}}\right]^{1/2},$$

$$g = \frac{b_B}{a + b_B} \equiv \frac{B\omega_0}{1 - \omega_0 + B\omega_0}, \quad (8)$$

where $\omega_0 = b/(a + b)$ is a single-scattering albedo.

The use of these assumptions leads to an accurate self-consistent version of the two-flow approximation.[6,7] This methodology improves significantly the Schwarzschild–Schuster[11,12] and the Kubelka–Munk[13] two-flow theories, especially for the case of seawater. Application of the above formula to the asymptotic diffuse attenuation is

$$\alpha_\infty = c((1 - \omega_0)\{1 - \omega_0(1 - 3B) + [B\omega_0(4 - 4\omega_0 + 9B\omega_0)]^{1/2}\})^{1/2}, \quad (9)$$

where $c = a + b$ is the beam attenuation coefficient. This equation is valid for all values of the backscattering probability B and the single-scattering albedo ω_0. The relative error of Eq. (9) does not exceed 5% (Ref. 7). It has been shown[7] that for a medium with strongly anisotropic scattering ($B \ll 0.02$) and arbitrary ω_0 or for a medium with $\omega_0 \ge 0.7$ and arbitrary B the accuracy of Eq. (9) approaches the accuracy obtained with numerical methods.

The derivation of Eqs. (2) does not imply that the inherent optical properties a and b_B are fixed.[14] The existence of a local depth regime that can be characterized by the angular radiance distribution given by Eqs. (4) (with the parameter $\bar{\mu}$ that varies with depth) is supported by Timofeyeva's experiments.[9] On this basis we assume that Eqs. (2) are also valid for the stratified sea.

3. Approximate Solutions for a Shallow Stratified Sea

Let $R(z)$ be the diffuse reflection coefficient of the sea layer enclosed between the depth z and the bottom. We denote $E_d^{\,0}$ as the irradiance of a horizontal surface placed just below the sea surface. In this case we can express the downward irradiance E_d through $E_d^{\,0}$ and the diffuse reflection coefficient R:

$$E_u(z) = R(z)E_d(z). \quad (10)$$

After inserting Eq. (10) into Eqs. (2) and rearranging terms, we obtain the following equation for diffuse reflection coefficient $R(z)$:

$$\frac{dR(z)}{dz} - 4[a(z) + b_B(z)]R(z) = -b_B(z)\{2 - \bar{\mu}(z) + R^2(z)[2 + \bar{\mu}(z)]\}. \quad (11)$$

Equation (11) for the diffuse reflection coefficient is of the Riccati type. This type of equation cannot be solved in analytic form for arbitrary functional dependencies $a(z)$ and $b_B(z)$.

A. Homogeneous Sea

For a homogeneous optically infinitely deep sea illuminated by diffuse light, $dR/dz = 0$ and the physical solution of Eq. (11) is reduced to $R = R_\infty$, where[7]

$$R_\infty = \left(\frac{1-\bar{\mu}}{1+\bar{\mu}}\right)^2 \equiv \left(\frac{\{a + 3b_B + [b_B(4a + 9b_B)]^{1/2}\}^{1/2} - \sqrt{a}}{\{a + 3b_B + [b_B(4a + 9b_B)]^{1/2}\}^{1/2} + \sqrt{a}}\right)^2. \quad (12)$$

This equation is valid for any combination of inherent optical properties a and b_B and satisfies two limit relations: at $\bar{\mu} \to 0$ (pure scattering and nonabsorbing medium) $R_\infty \to 1$ and at $\bar{\mu} \to 1$ (pure absorbing and nonscattering medium) $R_\infty \to 0$. The values of R_∞ given by Eq. (12) are close to the exact values for the diffuse reflection coefficient given in Ref. 15 for all ranges of variability of a and b_B.

Equation (12) also can be expressed in the form of Eq. (1) with the coefficient k equal to

$$k = \frac{1 + 4\bar{\mu}^2 - \bar{\mu}^4}{(1+\bar{\mu})^4}, \qquad 0.25 \le k \le 1. \quad (13)$$

Note that for the typical value of mean cosine $\bar{\mu} \sim 0.67$ the coefficient $k \approx 0.33$. This value agrees with the value proposed by Morel and Prieur.[16] When $b_B/a \ll 1$, Eq. (12) gives the value of R_∞ calculated in the single-scattering approximation $R_\infty = b_B/(4a)$. The values of R_∞, computed with Eq. (12) and values of $\bar{\mu}$ typical to the seawater, are close to those obtained by Gordon and Brown.[17]

Equation (11) also has another exact solution for a homogeneous shallow sea with depth z_B and Lambertian bottom albedo A_B:

$$R(z) = \frac{R_\infty + \xi(z)}{1 + R_0\xi(z)}, \quad (14)$$

where

$$R_0 = \frac{2+\bar{\mu}}{2-\bar{\mu}} R_\infty, \quad (15)$$

$$\xi(z) = \frac{A_B - R_\infty}{1 - R_0 A_B} \exp[-\nu(z_B - z)], \quad (16)$$

$$\nu = \frac{2a}{\bar{\mu}}\left(1 - \frac{\bar{\mu}^2}{1-g}\right) = a\,\frac{7 + 2\bar{\mu}^2 - \bar{\mu}^4}{\bar{\mu}(3-\bar{\mu}^2)}. \quad (17)$$

B. Stratified Sea

The next step is to obtain an approximate solution of Eq. (11) for a stratified sea. For open ocean areas and some coastal waters, $R^2(z) \le 5 \times 10^{-3}$, $\bar{\mu} \ll 1$, and the second term within braces on the right-hand side of Eq. (11) can be neglected. The result is a linear differential equation for R:

$$-\frac{dR(z)}{dz} + 4[a(z) + b_B(z)]R(z) = b_B(z)[2 - \bar{\mu}(z)]. \quad (18)$$

To solve this equation, let us introduce an auxiliary variable,

$$\tau(z) = 4\int_0^z [a(z) + b_B(z)]dz. \quad (19)$$

After implementation of Eq. (19) into Eq. (18) the following linear equation is obtained:

$$\frac{dR(\tau)}{d\tau} + R(\tau) = R_\infty(\tau), \quad (20)$$

where

$$R_\infty(\tau) = \left[\frac{1-\bar{\mu}(\tau)}{1+\bar{\mu}(\tau)}\right]^2 \approx \frac{2-\bar{\mu}(\tau)}{4}\,\frac{b_B(\tau)}{a(\tau) + b_B(\tau)}. \quad (21)$$

Here we neglected all second-order terms proportional to R_0^2 and R_∞^2. The solution of Eq. (20) is found in the form of partial plus general solutions: $R = R_p + R_g$, where $R_g = C_g \exp(-\tau)$. The partial solution is defined by Green's function $G(\tau)$ of Eq. (20):

$$R_p(\tau) = \int G(\tau - \tau')R_\infty(\tau')d\tau'. \quad (22)$$

Here

$$G(\tau) = H(\tau)\exp(-\tau) \quad (23)$$

is the solution of the following equation:

$$\frac{dG(\tau)}{d\tau} + G(\tau) = \delta(\tau), \quad (24)$$

where $\delta(\tau)$ is the Dirac delta function and $H(\tau)$ is the Heaviside or step function[18]:

$$\delta(\tau) = \begin{cases} \infty, & \tau = 0, \\ 0, & \tau \ne 0, \end{cases} \quad \int_{-\infty}^{+\infty} \delta(\tau)d\tau = 1,$$
$$H(\tau) = \begin{cases} 1, & \tau \ge 0, \\ 0, & \tau < 0, \end{cases} \quad \frac{dH(\tau)}{d\tau} = \delta(\tau). \quad (25)$$

We can now consider the sea with a shallow bottom. It is assumed that the bottom reflects light according to Lambert's law with an albedo A_B. In this case the boundary condition for $R(\tau)$ at the bottom depth $z = z_B$ is defined by

$$R(\tau|_{z=z_B}) = A_B. \quad (26)$$

Returning to the variable z (i.e., depth dependence) and applying the boundary condition in Eq. (26), we obtain a solution that defines the diffuse reflection coefficient for a vertically stratified shallow sea measured at depth z:

$$R(z) = 4\int_{z}^{z_B} R_\infty(z')\exp\left\{-4\int_{z}^{z'}[a(z'') + b_B(z'')]\mathrm{d}z''\right\} \times [a(z') + b_B(z')]\mathrm{d}z' + A_B \exp\left\{-4\int_{z}^{z_B}[a(z') + b_B(z')]\mathrm{d}z'\right\}. \quad (27)$$

Equation (27) defines the diffuse reflection coefficient of the vertically stratified shallow sea measured at depth z. By setting $z = 0$, we obtain the following equation that defines the diffuse reflection coefficient of the vertically stratified sea with depth z_B:

$$R = 4\int_{0}^{z_B} R_\infty(z)\exp\left\{-4\int_{0}^{z}[a(z') + b_B(z')]\mathrm{d}z'\right\}[a(z) + b_B(z)]\mathrm{d}z + A_B \exp\left\{-4\int_{0}^{z_B}[a(z') + b_B(z')]\mathrm{d}z'\right\}. \quad (28)$$

For the optically deep sea $\{\int_0^{z_B}[a(z) + b_B(z)]\mathrm{d}z \gg 1\}$ Eq. (28) can be simplified even further:

$$R_{(\infty)} = 4\int_{0}^{\infty} R_\infty(z)[a(z) + b_B(z)]\exp\left\{-4\int_{0}^{z}[a(z') + b_B(z')]\mathrm{d}z'\right\}\mathrm{d}z. \quad (29)$$

Equations (27) and (28) are derived for the case of diffuse illumination. The generalization to the case of combined illumination by Sun and sky can be achieved with the approach given in Ref. 19.

4. Validation

Several assumptions are usually made in the development of any two-flow theory. In none of these theories is it possible to write an analytically defined criterion for the validity of the obtained relationships. The accuracy of the results, therefore, was estimated by comparison of results to numerical solutions of the modeling of the transfer phenomenon.

The applicability of the self-consistent approach to waters with arbitrary values of a and b_B was investigated in Ref. 7. In this section we need to estimate only possible loss of precision that is due to the introduction of vertical inhomogeneity. To do this let us compare our results with the results obtained by use of Monte Carlo numerical computations of diffuse reflection coefficient $R_{(\infty)}$. We choose one of the least favorable cases of discontinuous stratification; then we compare the values of $R_{(\infty)}$ computed with Eq. (29) with the exact numerical results obtained by Gordon and Brown.[17] In Ref. 17 the Monte Carlo method is used to compute the diffuse reflection coefficient of a stratified sea. The stratification, used by Gordon and Brown, consists of two water layers. The thickness h and scattering coefficient b_1 of the upper layer are used as variable parameters. The lower semi-infinite layer has fixed inherent optical properties. The optical parameters taken for the lower layer are typical for the Sargasso Sea: $a = 0.062\ \mathrm{m}^{-1}$ and $b/a = 0.66$.[20] The same value of a and the same scattering phase function with $B = 0.0272$ are adopted for both layers.

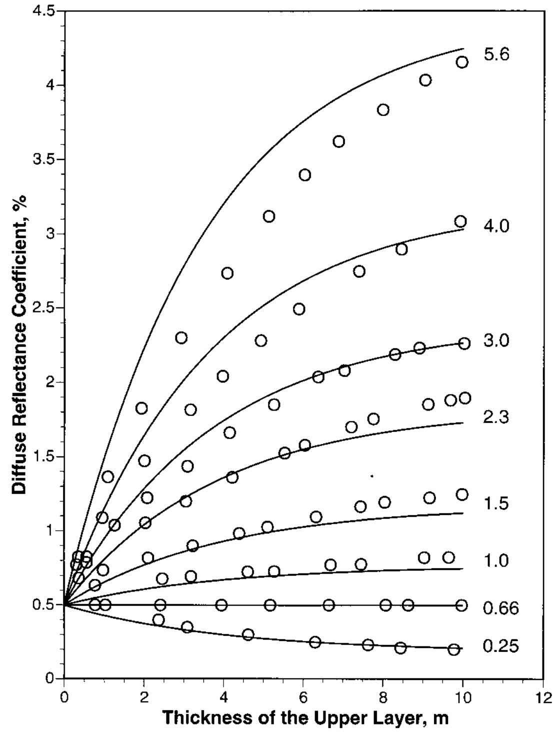

Fig. 1. Diffuse reflectance $R_{(2)}$ of the two-layer sea for various values of parameter b_B/a (numbers to the right-hand side of the curves). The solid curves represent values calculated with Eqs. (30)–(32), and the circles correspond to the values taken from Ref. 17.

Starting from Eq. (29), we derive a relation for the diffuse reflection of the two-layer sea. Let the upper layer of thickness h be described by the optical properties b_1, a_1, and B_1, and let the lower layer have the properties b_2, a_2, and B_2. By performing the integration in Eq. (29), we have

$$R_{(2)} = R_1 + (R_2 - R_1)\exp[-4h(a_1 + b_{B_1})], \quad (30)$$

where

$$R_1 = \left(\frac{1 - \bar{\mu}_1}{1 + \bar{\mu}_1}\right)^2, \qquad R_2 = \left(\frac{1 - \bar{\mu}_2}{1 + \bar{\mu}_2}\right)^2, \quad (31)$$

$$\bar{\mu}_1 \equiv \bar{\mu}(a_1, b_{B_1}), \qquad \bar{\mu}_2 \equiv \bar{\mu}(a_2, b_{B_2}). \quad (32)$$

Here $a_1 = a_2 = 0.062$, $b_{B_1} = B_1 b_1 = 0.0272\, b_1$, $b_{B_2} = B_2 b_2 = 0.0011\ \mathrm{m}^{-1}$, and $R_2 = 0.005$.

If we regard the values for $R_{(2)}$ taken from Gordon and Brown[17] as correct (see Fig. 1), then the relative error of computations with Eq. (30) does not exceed 15% for $\omega_0 \leq 0.85$.[21] Our additional simulations[21] show that in waters with the single-scattering albedo $\omega_0 \leq 0.6$ the error of calculations made with Eqs. (28) and (29) does not generally exceed 10%.

5. Conclusion

A set of equations, Eqs. (27)–(29), has been derived that can be used for computing the diffuse reflection coefficient of an arbitrary stratified sea. The precision of computations based on these formulas is comparable with the precision of *in situ* measurements of the diffuse reflection coefficient (10%–15%). These equations allow us to study the influence of the scattering layers in water on the structure of light upwelling from the sea. They can be used to calculate radiance contrasts between differently stratified areas of the sea. We can also use Eqs. (27)–(29) to retrieve vertical profiles of the sea optical properties by processing remotely measured optical spectral signatures.

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