One-parameter two-term Henyey–Greenstein phase function for light scattering in seawater

Vladimir I. Haltrin

A one-parameter two-term Henyey–Greenstein (TTHG) phase function of light scattering in seawater is proposed. The original three-parameter TTHG phase function was reduced to the one-parameter TTHG phase function by use of experimentally derived regression dependencies between integral parameters of the marine phase functions. An approach to calculate a diffuse attenuation coefficient in the depth of seawater is presented. © 2002 Optical Society of America

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1. Introduction

An analytic Henyey–Greenstein[1] (HG) scattering phase function is popular in radiative transfer calculations in astrophysics[2] and atmospheric and oceanic optics.[3,4] This phase function is convenient for some numeric and Monte Carlo calculations because it has an analytic form and expands to a simple Legendre polynomial series. It also allows for easy computation of asymptotic light radiance distribution. Unfortunately, the shape of the HG phase function inadequately represents the shape of realistic atmospheric and marine phase functions. For the case of atmospheric optics this shortcoming was successfully resolved by Kattawar[5] who proposed a two-term Henyey–Greenstein (TTHG) phase function. Unfortunately, Kattawar's TTHG phase function, in the form it was presented, is not directly applicable to marine waters. In this paper the similar approach is adopted but for oceanic water.

2. One-Parameter Two-Term Henyey–Greenstein Phase Function

The HG phase function[1,5,6] has the following analytic form:

$$p_{\mathrm{HG}}(\mu, g) = \frac{1 - g^2}{(1 - 2g\mu + g^2)^{3/2}}, \qquad \mu = \cos\vartheta, \quad (1)$$

where ϑ is a scattering angle, and an asymmetry parameter g is equal to the average cosine $\overline{\cos\vartheta}$ over the angular distribution in Eqs. (1) defined as

$$\overline{\cos\vartheta} = \frac{1}{2}\int_{-1}^{1} p(\mu)\mu \mathrm{d}\mu, \qquad \frac{1}{2}\int_{-1}^{1} p(\mu)\mathrm{d}\mu = 1. \quad (2)$$

The HG phase function in Eqs. (1) has the following expansion into the Legendre polynomial series:

$$p_{\mathrm{HG}}(\mu, g) = \sum_{n=0}^{\infty} (2n + 1)g^n P_n(\mu), \qquad 0 \le g \le 1. \quad (3)$$

The average of the squared cosine for the HG phase function of Eqs. (1) is

$$\overline{\cos^2\vartheta} = \frac{1}{2}\int_{-1}^{1} p_{\mathrm{HG}}(\mu)\mu^2 \mathrm{d}\mu = \frac{1}{3} + \frac{2}{3}g^2, \quad (4)$$

and the probability of backscattering (equal to the ratio of backscattering coefficient b_b to the total scattering coefficient b, $B \equiv b_b/b$) is defined by the following equation:

$$\begin{aligned} B|_{\mathrm{HG}} &= \frac{1}{2}\int_{-1}^{0} p_{\mathrm{HG}}(\mu, g)\mathrm{d}\mu \\ &\equiv \frac{1}{2}\int_{0}^{1} p_{\mathrm{HG}}(-\mu, g)\mathrm{d}\mu \\ &= \frac{1 - g}{2g}\left[\frac{1 + g}{(1 + g^2)^{1/2}} - 1\right]. \end{aligned} \quad (5)$$

The HG phase function has remarkable analytical properties presented by Eqs. (1) and (3). At the same time it is ill-suited to model scattering in real

V. I. Haltrin (haltrin@nrlssc.navy.mil) is with the Naval Research Laboratory, Ocean Optics Section, Code 7333, Stennis Space Center, Mississippi 39529-5004.

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seawater because the shape of the HG phase function significantly differs from the shape of real oceanic phase functions.[6,7] The HG phase function imitates only the first, forward-directed peak of the experimental phase function and ignores the second one, the backward-directed peak.

To overcome this shortcoming and retain all desirable properties of a HG phase function, a TTHG phase function was chosen in a form proposed by Kattawar:[5]

$$p_{\mathrm{TTHG}}(\mu, \alpha, g, h) = \alpha p_{\mathrm{HG}}(\mu, g) + (1 - \alpha) \times p_{\mathrm{HG}}(\mu, -h),\quad 0 \le \alpha, g, h \le 1. \tag{6}$$

Here α is a weight of the forward-directed HG phase function, $(1 - \alpha)$ is a weight of the backward-directed HG phase function, g is an asymmetry factor of the forward-directed HG phase function, and h is an asymmetry factor of the backward-directed HG phase function. The sign before the positive asymmetry factors determines the direction of elongation: A positive sign determines elongation forward and a negative sign determines elongation backward. So Eqs. (6) define the phase function that has two peaks: the forward peak with the weight α and asymmetry g and the backward peak with the weight $(1 - \alpha)$ and asymmetry h. The TTHG phase function of Eqs. (6) has the following expansion into the Legendre polynomial series:

$$p_{\mathrm{TTHG}}(\mu, \alpha, g, h) = \sum_{n=0}^{\infty} (2n + 1)[\alpha g^n + (1 - \alpha) \times (-h)^n] P_n(\mu). \tag{7}$$

The integral parameters, which include backscattering probability B, average cosine $\overline{\cos\vartheta}$, and average square of cosine $\overline{\cos^2\vartheta}$, of the TTHG phase function of scattering are given by the following equations:

$$B \equiv \frac{1}{2}\int_{-1}^{0} p_{\mathrm{TTHG}}(\mu, g)\mathrm{d}\mu = \alpha\frac{1 - g}{2g}\left[\frac{1 + g}{(1 + g^2)^{1/2}} - 1\right] + (1 - \alpha)\frac{1 + h}{2h}\left[1 - \frac{1 - h}{(1 + h^2)^{1/2}}\right], \tag{8}$$

$$\overline{\cos\vartheta} = \frac{1}{2}\int_{-1}^{1} p_{\mathrm{TTHG}}(\mu)\mu\mathrm{d}\mu = \alpha g + (1 - \alpha)(-h) \equiv \alpha(g + h) - h, \tag{9}$$

$$\overline{\cos^2\vartheta} = \frac{1}{2}\int_{-1}^{1} p_{\mathrm{TTHG}}(\mu)\mu^2\mathrm{d}\mu = \frac{1}{3} + \frac{2}{3}[\alpha(g^2 - h^2) + h^2]. \tag{10}$$

To reduce a number of independent parameters (α, g, h) in Eqs. (6) for the TTHG phase function, I used experimental data presented in the paper by Timofeyeva,[8] in which a number of regression relationships between integral parameters of experimentally measured phase functions are given. By using the data published by Timofeyeva, I rederived the regression relationships between integral characteristics of the phase function in a form convenient for the elimination of the two extra parameters in Eqs. (6). These new regressions are represented as follows:

$$\overline{\cos\vartheta} = 2\frac{1 - 2B}{2 + B},\qquad r^2 \cong 0.99999, \tag{11}$$

$$\overline{\cos^2\vartheta} = \frac{6 - 7B}{3(2 + B)},\qquad r^2 \cong 0.99999. \tag{12}$$

The original measurements by Timofeyeva support Eqs. (11) and (12) in the range of experimental data, i.e. $0.05 \le B \le 0.25$. This range encompasses the variability of B's corresponding to seawater; however, because of the specifics of the derivations,[9] Eqs. (11) and (12) are valid in the whole range of variability of backscattering probability: $0 \le B \le 0.5$. The relationships of Eqs. (11) and (12) should be considered as a better alternative to the original regressions by Timofeyeva. They are better because they give values of parameters that both lie in the range of experimental error for $0.05 \le B \le 0.25$ and satisfy two asymptotic conditions at delta-shaped scattering ($B = 0$) and isotropic scattering ($B = 0.5$). Equations (11) and (12) also give values almost identical to the values computed with Timofeyeva's original regressions.

By solving the relationships of Eqs. (11) and (12) with Eqs. (8)–(10), we obtain the following coupling relationships between parameters (α, h) and the first asymmetry parameter g:

$$\alpha = \frac{h(1 + h)}{(g + h)(1 + h - g)}, \tag{13}$$

$$h = -0.3061446 + 1.000568g - 0.01826332g^2 + 0.03643748g^3,\qquad 0.30664 < g \le 1. \tag{14}$$

Substitution of Eqs. (13) and (14) into Eqs. (6) or Eq. (7) gives us a one-parameter TTHG phase function of light scattering in seawater with integral parameters (B, $\overline{\cos\vartheta}$, $\overline{\cos^2\vartheta}$) adjusted to the experimentally derived relationships given by Eqs. (11) and (12). The dependence of integral parameters (B, $\overline{\cos\vartheta}$, $\overline{\cos^2\vartheta}$) of the marine TTHG phase function on the asymmetry parameter g is shown in Fig. 1.

By solving a radiative transfer equation in the depth of seawater, as demonstrated in Appendix A, we can obtain an expression for the eigenvalue of this equation as a function of probability of scattering $B \equiv b_b/b$ and single-scattering albedo $\omega_0 = b/c$ (here c is a total beam attenuation coefficient of seawater). Because, by definition, the eigenvalue is a ratio of the diffuse attenuation coefficient to the beam attenuation coefficient, we have the following simple formula

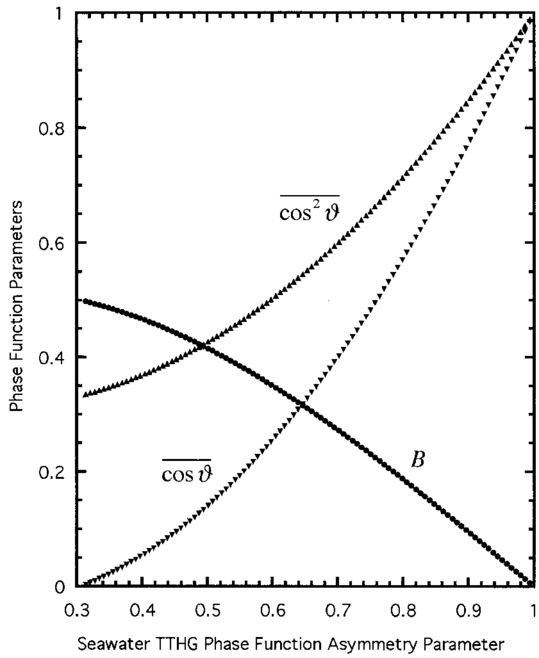

Fig. 1. Dependence of integral parameters (B, $\overline{\cos \vartheta}$, $\overline{\cos^2 \vartheta}$) of the marine TTHG phase function on asymmetry parameter g.

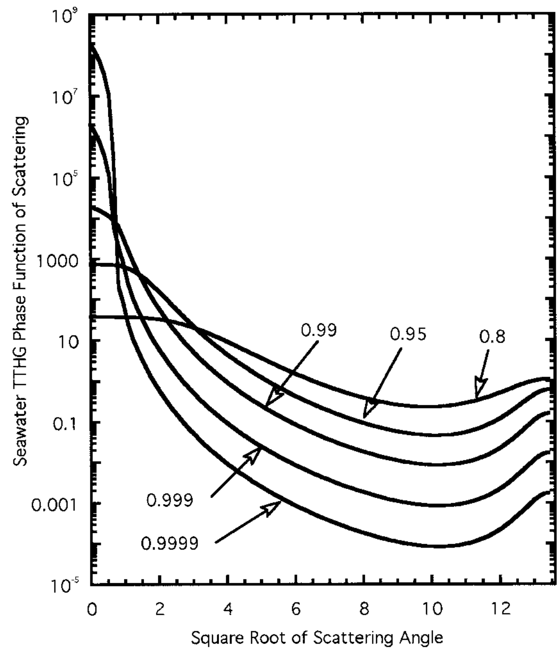

Fig. 2. Samples of marine TTHG phase functions for different values of asymmetry parameter g.

for the diffuse attenuation coefficient of light in the depth of seawater:

$$k_\infty = c\gamma(B, \omega_0), \tag{15}$$

with the eigenvalue $\gamma(B, \omega_0)$ defined by Eq. (A15) of Appendix A.

3. Connection of the Seawater Two-Term Henyey–Greenstein Phase Function with the Beam Scattering Coefficient

The seawater TTHG phase function proposed above depends on only one parameter g. Experimental data published by Petzold[7] allow us to derive the following relationships that connect the seawater backscattering coefficient b_b and the seawater backscattering probability $B = b_b/b$ with the seawater scattering coefficient b at 515 nm:

$$b_b = 0.000977 + 0.01(b - 0.001954)\,\frac{0.962 + 0.532b}{1 + 0.01064b}, \quad r^2 = 0.993, \tag{16}$$

$$B \equiv \frac{b_b}{b} = \frac{0.000977}{b} + 0.01 \times \left(1 - \frac{0.001954}{b}\right)\frac{0.962 + 0.532b}{1 + 0.01064b}, \quad \lambda = 515\ \text{nm}. \tag{17}$$

When we use Eqs. (8), (13), and (14) with Eqs. (16) and (17), it is easy to obtain the following relationship that couples parameter g in Eqs. (7)–(14) with the beam scattering coefficient:

$$g = 1 - \frac{0.001247\ \text{m}^{-1}}{b_{515\ \text{nm}}}, \quad 0.001954\ \text{m}^{-1} \le b_{515\ \text{nm}} \le 2\ \text{m}^{-1}, \quad r^2 = 0.985. \tag{18}$$

Equations (18) should be used in seawater radiative transfer calculations to match the elongation properties of the seawater TTHG phase function of Eqs. (6), (13), and (14) with the beam attenuation coefficient. The generalization to other wavelengths of visible spectrum can be accomplished with the approach proposed in Ref. 10 and implemented as an algorithm and a FORTRAN code in Ref. 11.

4. Examples of Seawater Two-Term Henyey–Greenstein Phase Functions and Comparison with Experimental Data

The seawater one-parameter TTHG phase function, being slightly more complex than the simple HG phase function, has the following important advantage over the HG phase function: Similar to the realistic phase functions, it has two prominent peaks, the largest one in the forward direction and the smallest one in the backward direction. The simple HG phase function has only one peak, which is in the forward direction.

Figure 2 shows samples of seawater TTHG phase functions for typical values of parameter g, and Fig. 3 shows comparisons among the experimental Petzold phase function, the HG phase function, and the one-parameter seawater TTHG phase function.

The theory presented in Appendix A shows that it is possible to derive some optical properties and pa-

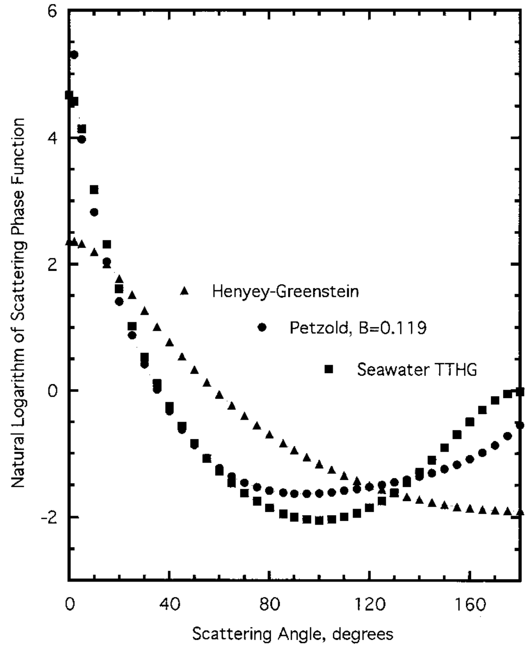

Fig. 3. Comparison of the HG phase function, the one-parameter seawater TTHG phase function, and the experimental Petzold phase function[7] with the same value of backscattering probability $B = 0.119$.

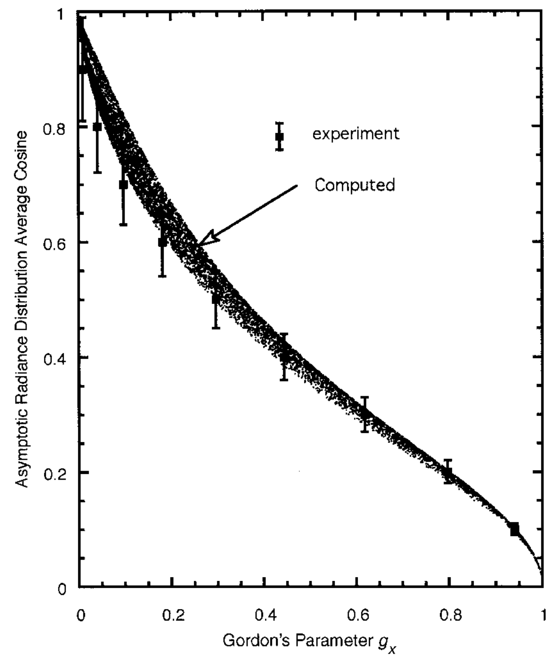

Fig. 4. Comparison of average cosines of the asymptotic radiance distribution computed for all possible combinations of B and ω_0 with the experimental average cosines plotted against Gordon's parameter $g_x = B\omega_0/(1 - \omega_0 + B\omega_0)$.

rameters of the light field with only an analytic approach. One of the most important and easiest to derive parameters of the light field in the depth of scattering medium is an average cosine: $\bar{\mu} = a/k_\infty \equiv (1 - \omega_0)/\gamma(B, \omega_0)$ with $\gamma(B, \omega_0)$ given by Eq. (A15) and a being an absorption coefficient of seawater.

To compare results of modeling with the seawater TTHG phase function with the experimental data, a set of random pairs of backscattering probability and single-scattering albedos (B, ω_0) was generated. These pairs, being homogeneously distributed in the rectangle ($0 \le B \le 0.5$, $0 \le \omega_0 \le 1$), simulate samples of all possible values of (B, ω_0). For each pair of (B, ω_0), a value of an average cosine $\bar{\mu}$ was computed. The results of the simulated $\bar{\mu}$'s and the values of the experimentally measured average cosines[12] are plotted against Gordon's parameter $g_x = B\omega_0/(1 - \omega_0 + B\omega_0)$ (see Fig. 4). In spite of the fact that, in reality, the values of (B, ω_0) are not independent but correlate with each other[10,13] and the seawater TTHG phase function of scattering does not ideally match a realistic one (see Fig. 3), the results of the simulation are in a good agreement with the experimental data. This simulation gives an additional argument that the proposed phase function of scattering is a good candidate for use in seawater radiative transfer computations, especially in those cases when the analytical properties of HG phase function are important.[14]

5. Conclusion

It was shown that a realistic marine phase function can be represented in the form of a one-parameter two-term Henyey–Greenstein phase function with the same dependencies between integral characteristics of the phase function. In some special computational cases,[14] this phase function may be more preferable than the more realistic Fournier–Forand[15] phase function or an analytic form of Petzold phase functions.[6] The proposed phase function may be convenient for radiative transfer modeling.[3,16–19]

Appendix A: Eigenvalues and Radiance Distribution of Light in an Asymptotic Regime

It is easy to show that the solution of the radiative transfer equation in the asymptotic regime (when the optical depth is much larger than 1) is azimuthally symmetric and can be written as

$$L(\tau, \mu) = L_0\Psi(\mu)\exp(-\gamma\tau), \tag{A1}$$

where $L(\tau, \mu)$ is the radiance of light at optical depth $\tau = \int_0^z c(z')\mathrm{d}z'$, $c(z)$ is a beam attenuation coefficient at physical depth z (for homogeneous water $\tau = cz$), L_0 is determined from the boundary conditions (described, for example, in Refs. 3, 5, 16, 17, 20, 21), $\psi(\mu)$ is a radiance distribution as a function of $\mu = \cos\theta$, θ is the zenith angle, $c = a + b$, a and b are absorption and scattering coefficients, and γ is the eigenvalue of the asymptotic radiative transfer equation. For homogeneous water $\gamma = k_\infty/c$, where k_∞ is the asymptotic diffuse attenuation coefficient.

The asymptotic radiance distribution $\Psi(\mu)$ in Eq. (A1) is a solution of the following characteristic or asymptotic equation for the radiative transfer:

$$(1 - \gamma\mu)\Psi(\mu) = \frac{\omega_0}{2}\int_{-1}^{1} \Psi(\mu')\bar{p}(\mu, \mu')\mathrm{d}\mu', \tag{A2}$$

where $\omega_0 = b/c$ is the albedo for single scattering, $\mu' = \cos \theta'$, and

$$\bar{p}(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} p(\cos \vartheta) d\phi,$$

$$\left.\cos \vartheta = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos(\phi - \phi')\right\} \tag{A3}$$

is the phase function averaged over azimuth angle ϕ, ϑ is the scattering angle, and the scattering phase function $p(\cos \vartheta)$ is normalized by the following condition:

$$\frac{1}{2} \int_0^{\pi} p(\cos \vartheta) \sin \vartheta d\vartheta = 1. \tag{A4}$$

If we represent the phase function in Eq. (A3) in the form of the Legendre polynomial series,

$$p(\cos \vartheta) = \sum_{n=0}^{\infty} s_n P_n(\cos \vartheta), \qquad s_0 = 1, \tag{A5}$$

where $P_n(\cos \vartheta)$ is the Legendre polynomial of the nth order, then we have

$$\bar{p}(\mu, \mu') = \sum_{n=0}^{\infty} s_n P_n(\mu) P_n(\mu'). \tag{A6}$$

The Legendre polynomial series can also be used to represent the radiance distribution $\Psi(\mu)$:

$$\Psi(\mu) = \sum_{n=0}^{\infty} (2n + 1) \Psi_n P_n(\mu). \tag{A7}$$

By inserting Eqs. (A6) and (A7) into Eq. (A2) and utilizing the recursion relation between Legendre polynomials,

$$(2n + 1)\mu P_n(\mu) = n P_{n-1}(\mu) + (n + 1) P_{n+1}(\mu), \tag{A8}$$

we obtain the following relationships between the expansion coefficients ψ_n and s_n (Refs. 5, 6, 22):

$$\left.\begin{array}{c} \psi_0 - \gamma\psi_1 = \omega_0 \psi_0 s_0, \\ \gamma(n + 1)\psi_{n+1} - (2n + 1 - \omega_0 s_n)\psi_n + \gamma n \psi_{n-1} = 0, \\ n = 1, \ldots, \infty \end{array}\right\}. \tag{A9}$$

From the condition of solvability of the homogeneous system of Eqs. (A9), we can find the connection among γ, ω_0, and s_n. The condition is that the determinant Δ of the system of Eqs. (A9) is equal to zero:

$$\Delta = \begin{vmatrix} 1 - \omega_0 & -\gamma & 0 & 0 & \cdots \\ -\gamma & 3 - \omega_0 s_1 & -2\gamma & 0 & \cdots \\ 0 & -2\gamma & 5 - \omega_0 s_2 & -3\gamma & \cdots \\ 0 & 0 & -3\gamma & 7 - \omega_0 s_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{A10}$$

As was shown in Ref. 23, Eq. (A10) is equivalent to the following infinite chain fraction equation:

$$1 - \omega_0 = \cfrac{\gamma^2}{3 - \omega_0 s_1 - \cfrac{4\gamma^2}{5 - \omega_0 s_2 - \cfrac{9\gamma^2}{7 - \omega_0 s_3 - \cdots}}}, \tag{A11}$$

or

$$1 - \omega_0 = \Delta_1, \tag{A12}$$

where Δ_n can be obtained from the following recursion:

$$\Delta_n = \frac{(n\gamma)^2}{2n + 1 - \omega_0 s_n - \Delta_{n+1}}, \qquad n = 1, \ldots, \infty. \tag{A13}$$

Table 1. Eigenvalues γ of the Asymptotic Equation for Transfer for the One-Parameter Seawater TTHG Scattering Phase Function for Different Values of the Single-Scattering Albedo ω_0 and the Probability of Backscattering B

	Eigenvalue								
B/ω_0	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.005	0.862038	0.716817	0.670190	0.521512	0.447405	0.323231	0.222598	0.120107	0.021534
0.050	0.860399	0.778234	0.692595	0.603911	0.512142	0.416768	0.316303	0.205647	0.060727
0.100	0.893488	0.823050	0.747403	0.667069	0.581793	0.490387	0.389641	0.269236	0.084465
0.150	0.919372	0.859252	0.792536	0.719673	0.640036	0.551622	0.449466	0.318981	0.102089
0.200	0.940713	0.890095	0.831655	0.765632	0.690945	0.604749	0.500527	0.360395	0.116453
0.250	0.958529	0.916864	0.866245	0.806596	0.736330	0.651806	0.545203	0.396069	0.128689
0.300	0.973240	0.940149	0.897057	0.843465	0.777248	0.694039	0.584946	0.427481	0.139393
0.350	0.984925	0.960157	0.924468	0.876781	0.814398	0.732298	0.620730	0.455566	0.148921
0.400	0.993398	0.976777	0.948583	0.906861	0.848265	0.767197	0.653247	0.480965	0.157513
0.450	0.998330	0.989516	0.969196	0.933815	0.879178	0.799195	0.683010	0.504138	0.165335
0.500	0.999909	0.997414	0.985624	0.957504	0.907332	0.828635	0.710412	0.525430	0.172511

Table 2. Coefficients $\gamma_n(\omega_0)$ for Eq. (A15)

ω_0	$\gamma_0(\omega_0)$	$\gamma_1(\omega_0)$	$\gamma_2(\omega_0)$	$\gamma_3(\omega_0)$	$\gamma_4(\omega_0)$	$\gamma_5(\omega_0)$	r^2
0.05	0.95609	0.41762	1.8653	5.3139	9.0900	6.6281	0.99992
0.10	0.90763	0.73428	3.2760	10.415	18.715	13.196	0.99991
0.15	0.85958	0.97587	4.1853	13.523	24.053	16.382	0.99995
0.20	0.81145	1.1718	4.7173	14.839	25.423	16.520	0.99996
0.25	0.76304	1.3459	5.1578	15.933	26.616	16.805	0.99996
0.30	0.71437	1.5039	5.5563	17.038	28.177	17.665	0.99997
0.35	0.66548	1.6481	5.9213	18.146	30.012	18.943	0.99997
0.40	0.61639	1.7788	6.2421	19.151	31.795	20.297	0.99998
0.45	0.56712	1.8959	6.5105	19.966	33.270	21.463	0.99998
0.50	0.51768	1.9995	6.7251	20.552	34.304	22.303	0.99998
0.55	0.46807	2.0895	6.8909	20.910	34.870	22.774	0.99998
0.60	0.41827	2.1657	7.0186	21.070	35.006	22.899	0.99998
0.65	0.36826	2.2278	7.1237	21.092	34.810	22.741	0.99999
0.70	0.31802	2.2744	7.2282	21.069	34.447	22.413	0.99999
0.75	0.26751	2.3030	7.3619	21.149	34.188	22.102	0.99999
0.80	0.21670	2.3080	7.5606	21.546	34.462	22.112	0.99999
0.85	0.16560	2.2753	7.8495	22.521	35.857	22.886	0.99999
0.90	0.11433	2.1659	8.1620	24.143	38.785	24.816	0.99998
0.95	0.063265	1.8473	7.9291	24.867	41.055	26.660	0.99996
0.97	0.042914	1.5580	7.1137	22.981	38.530	25.250	0.99993
0.99	0.021309	0.97834	4.7437	15.778	26.869	17.777	0.99989

If we know the values of single-scattering albedo ω_0 and phase function coefficients s_n, the eigenvalue γ can be obtained numerically when we solve Eqs. (A12) and (A13). In this case we compute all coefficients ψ_n of the asymptotic radiance distribution by using the following relationships:

$$\left.\begin{aligned} &\psi_0 = 1, \qquad \psi_1 = \frac{1-\omega_0}{\gamma}, \\ &\psi_n = \left(2 - \frac{1+\omega_0 s_{n-1}}{n}\right)\frac{\psi_{n-1}}{\gamma} - \\ &\quad \left(1-\frac{1}{n}\right)\psi_{n-2}, \qquad n = 2, \ldots, \infty \end{aligned}\right\} \tag{A14}$$

and therefore restore the radiance distribution $\Psi(\mu)$ given by Eq. (A6).

The eigenvalue solutions for the seawater TTHG phase function for the pairs of input parameters (ω_0, B), computed numerically, are given in Table 1. Using a largely expanded version of this table it is possible to derive the following equation that connects the eigenvalue γ with the single-scattering albedo and backscattering probability (ω_0, B):

$$\gamma(B, \omega_0) = \gamma_0(\omega_0) - \sum_{n=1}^{5} \gamma_n(\omega_0)(-1)^n B^n. \tag{A15}$$

The coefficients $\gamma_n(\omega_0)$, $n = 0, \ldots, 5$, in Eq. (A15) are given in Table 2.

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